Motivation: Dynamical Systems

There are two broad (and interwoven) kinds of dynamical systems: discrete and continuous.

Discrete dynamics: State space S, $x_0 \in S$
Function $f: S \to S$

Study the behaviour of the sequence

$$x_{n+1} = f(x_n)$$

Continuous dynamics: State space S ($= \mathbb{R}^d$ or a manifold), $x_0 \in S$
Function ("vector field") $f: S \to S$ (or maybe $T(S)$)

Study the behaviour of the curve

$$\frac{d}{dt} x(t) = f(x(t)), \quad x(0) = x_0.$$

In both settings, the dynamics is **deterministic**: if you know f and x_0 with perfect precision, you know the full dynamics (in principle).

But what if there's noise / measurement error?

A Model for Noisy (Discrete) Dynamics

State space S. We want to iterate $x_{n+1} = f(x_n)$ as before, but this time with the function changing randomly as well.

That's too vague/broad to do any consistent analysis. Here's a more precise model:

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, measurable space $(R, \mathcal{G})$

i.i.d. sequence $\xi_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (R, \mathcal{G})$

State (measurable) space $(S, \mathcal{B})$

Function: $f : S \times R \to S$

Dynamical System:

$X_{n+1} =$

That is: our random function has the form

$X_n = f(X_{n-1}, \xi_n) = f(f(X_{n-2}, \xi_{n-1}), \xi_n)$

Set $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$, $n \geq 0$

$= \sigma(X_0, \xi_1, \dots, \xi_n)$

$\therefore$ ξ_{n+1} is independent from $\mathcal{F}_n$

Thus, if $g \in \mathbb{B}(S, \mathcal{B})$,

$$\mathbb{E}[g(X_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}[g(f(X_n, \xi_{n+1})) \mid \mathcal{F}_n]$$

Set $Q_n(x, \cdot) := \mathrm{Law}(f(x, \xi_n))$, L_n its Markov generator.

Thus $\mathbb{E}[g(X_{n+1}) \mid \mathcal{F}_n]$

$$= \mathbb{E}[g(X_{n+1}) \mid X_0, X_1, \dots, X_n] = (L_{n+1}g)(X_n)$$

Take $\mathbb{E}_{\sigma(X_n)}$ of both sides.

$$\mathbb{E}_{\sigma(X_n)}[\mathbb{E}[g(X_n) \mid \mathcal{F}_n]]$$

$$\mathbb{E}_{\sigma(X_n)}[(L_{n+1}g)(X_n) \mid \sigma(X_n)]$$

we've arrived at the Markov Property:

$$\mathbb{E}[g(X_{n+1}) \mid X_0, \dots, X_n] = \mathbb{E}[g(X_{n+1}) \mid X_n] \text{ a.s. } \forall g \in B(S, \mathcal{B})$$

Iterating the above argument yields the nominally stronger

$$\boxed{\mathbb{E}[g(X_n) \mid \mathcal{F}_k] = \mathbb{E}[g(X_n) \mid X_k] \text{ a.s. } \forall k < n.}$$