Motivation: Dynamical Systems

There are two broad (and interwoven) kinds of dynamical systems: discrete and continuous.

Discrete dynamics: State space S, $x_0 \in S$
Function $f : S \to S$

Study the behaviour of the sequence

$$x_{n+1} = f(x_n)$$

Continuous dynamics: State space S ($= \mathbb{R}^d$ or a manifold), $x_0 \in S$
Function ("vector field") $f : S \to S$ (or maybe $T(S)$)

Study the behaviour of the curve

$$\frac{d}{dt} x(t) = f(x(t)), \quad x(0) = x_0 .$$

In both settings, the dynamics is **deterministic**: if you know f and x_0 with perfect precision, you know the full dynamics (in principle).

But what if there's noise / measurement error?

A Model for Noisy (Discrete) Dynamics

State space S. We want to iterate $x_{n+1} = f_n(x_n)$ as before, but this time with the function changing randomly as well. $f_n : S \to S$ random function.

That's too vague / broad to do any consistent analysis. Here's a more precise model:

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, measurable space $(R, \mathcal{G})$

i.i.d. sequence $\xi_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (R, \mathcal{G})$

State (measurable) space $(S, \mathcal{B})$

measurable Function: $f : S \times R \to S$

Dynamical System:

$$X_{n+1} = f(X_n, \xi_{n+1}) \qquad \text{with } X_0 : (\Omega, \mathcal{F}) \to (S, \mathcal{B}) \text{ measurable}$$

indep. from $\{\xi_n\}_{n=1}^{\infty}$

That is: our random function has the form $f_n = f(\cdot, \xi_{n+1})$

$$X_n = f(X_{n-1}, \xi_n) = f(f(X_{n-2}, \xi_{n-1}), \xi_n)$$

$$= F_n(X_0, \xi_1, \xi_2, \cdots, \xi_n)$$

$$F_n : S \times \mathbb{R}^n \to S \qquad F_n(x, y_1, \cdots, y_n)$$

$$= f(f(\cdots f(f(x, y_1), y_2) \cdots), y_{n-1}), y_n)$$

measurable $\mathcal{B} \otimes \mathcal{B}^{\otimes n} / \mathcal{B}$.

Set $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, $n \geq 0$ $\qquad \mathcal{F}_0 = \sigma(X_0)$

$$= \sigma(X_0, \xi_1, \cdots, \xi_n)$$

$\therefore$ ξ_{n+1} is independent from $\mathcal{F}_n$ (grouping)

Thus, if $g \in \mathbb{B}(S, \mathcal{B})$,

$$E[g(X_{n+1}) \mid \mathcal{F}_n] = E[g(f(X_n, \xi_{n+1})) \mid \mathcal{F}_n] = E[g \circ f(x, \xi_{n+1})] \Big|_{x = X_n}$$

Set $Q_n(x, \cdot) := \text{Law}(f(x, \xi_n))$, L_n its Markov generator.

$$= \int g(y)\, Q_{n+1}(x, dy) \Big|_{x = X_n}$$

$$= (L_{n+1} g)(X_n)$$

Thus $\mathbb{E}[g(X_{n+1}) \mid \mathcal{F}_n]$ $\quad \mathbb{B}(\Omega, \sigma(X_n))$

$$= \mathbb{E}[g(X_{n+1}) \mid X_0, X_1, \dots, X_n] = (L_{n+1}g)(X_n)$$

Take $\mathbb{E}_{\sigma(X_n)}$ of both sides.

$$\mathbb{E}_{\sigma(X_n)}(\mathbb{E}[g(X_{n+1}) \mid \mathcal{F}_n]) = \mathbb{E}[g(X_{n+1}) \mid X_n] \text{ a.s. b/c } \sigma(X_n) \subseteq \mathcal{F}_n.$$

$$\mathbb{E}_{\sigma(X_n)}[(L_{n+1}g)(X_n) \mid \sigma(X_n)] = L_{n+1}g(X_n) = \mathbb{E}[g(X_{n+1}) \mid \mathcal{F}_n] \text{ a.s.}$$

we've arrived at the Markov Property:

$$\mathbb{E}[g(X_{n+1}) \mid X_0, \dots, X_n] = \mathbb{E}[g(X_{n+1}) \mid X_n] \text{ a.s.} \quad \forall g \in \mathbb{B}(S, \mathcal{B})$$

Iterating the above argument yields the nominally stronger

$$\boxed{\mathbb{E}[g(X_n) \mid \mathcal{F}_k] = \mathbb{E}[g(X_n) \mid X_k] \text{ a.s. } \forall k < n.}$$

$$\mathbb{E}[g(X_n) \mid \mathcal{F}_k] = (L_n L_{n-1} \cdots L_{k+1} g)(X_k).$$