Given a probability kernel over $(S_1, \mathcal{B}_1), (S_2, \mathcal{B}_2)$

$$Q : S_1 \times \mathcal{B}_2 \to [0,1]$$

we can integrate bounded measurable functions:

$$\int f(y)\, Q(x, dy), \quad f \in \mathbb{B}(S_2, \mathcal{B}_2).$$

and the result is a new measurable function of $x \in S_1$.
This function is also bounded:

$$\left| \int f(y)\, Q(x, dy) \right|$$

Def: Equip $\mathbb{B}(S_2, \mathcal{B}_2)$ with the norm $\|f\|_\infty := \sup |f|$.
For $f \in \mathbb{B}(S_2, \mathcal{B}_2)$, define $L_Q f \in \mathbb{B}(S_1, \mathcal{B}_1)$ by

Thus $L_Q : \mathbb{B}(S_2, \mathcal{B}_2) \to \mathbb{B}(S_1, \mathcal{B}_1)$. We call this operator a **Markov generator.**

Prop: Given a probability kernel Q over $(S_1, \mathcal{B}_1), (S_2, \mathcal{B}_2)$ the associated Markov generator

$$L_Q : \mathbb{B}(S_2, \mathcal{B}_2) \longrightarrow \mathbb{B}(S_1, \mathcal{B}_1)$$

has the following properties.

0. L_Q is a linear operator.
1. $L_Q(\mathbb{1}) = \mathbb{1}$
2. $L_Q(f) \geq 0$ if $f \geq 0$.
3. If $f_n \in \mathbb{B}(S_2, \mathcal{B}_2)$ and $f_n \to f$ boundedly, then $L_Q f_n \to L_Q f$ boundedly.

Pf. 0.

1.

2.

3.

Note: If $L : \mathbb{B}(S_2, \mathcal{B}_2) \to \mathbb{B}(S_1, \mathcal{B}_1)$ satisfies

0. L is linear, and
2. $Lf \geq 0$ if $f \geq 0$, then it follows that

$$f \leq g \Rightarrow Lf \leq Lg$$

$$\therefore \quad |Lf| \leq L|f|$$

And if 1. $L(1) = 1$ holds, then $f \leq \|f\|_\infty$

$$\Rightarrow |Lf|(x) \leq (L|f|)(x) \leq L(\|f\|_\infty)$$

If $f \in \mathbb{B}_{\mathbb{C}}(S_2, \mathcal{B}_2)$, we still have $|Lf| \leq L(|f|)$:

$$(Lf)(x) = |Lf(x)| e^{i\theta_x}$$

$$\therefore \quad |Lf(x)| = (Lf)(x) e^{-i\theta_x}$$

Prop: Let $L : \mathbb{B}(S,\mathcal{B}) \to \mathbb{B}(S,\mathcal{B})$ be a linear operator satisfying 1-3:

1. $L(1) = 1$
2. $L(f) \geq 0$ if $f \geq 0$.
3. If $f_n \to f$ boundedly in $\mathbb{B}(S,\mathcal{B})$, then $L(f_n) \to L(f)$ boundedly in $\mathbb{B}(S,\mathcal{B})$.

Then $Q(x,B) := L(\mathbb{1}_B)(x)$ defines a probability kernel over $(S,\mathcal{B})^2$, and $L_Q = L$.

Pf. As noted on the last slide, 1-2 $\Rightarrow$ $Lf \leq Lg$ when $f \leq g$.

$$0 \leq \mathbb{1}_B \leq 1$$

$$B = \bigsqcup_{k=1}^{\infty} B_k \Rightarrow \mathbb{1}_B = \sum_{k=1}^{\infty} \mathbb{1}_{B_k}. \text{ Set } f_n = \sum_{k=1}^{n} \mathbb{1}_{B_k}$$

Thus $Q(x,B) := L(\mathbb{1}_B)(x)$ defines a prob. kernel.

We've left to show that $L = L_Q$: i.e. want

$$L(f) = L_Q(f) = \int f(y)\, Q(\cdot, dy)$$

True for $f = \mathbb{1}_B$:

Now use Dynkin's multiplicative systems theorem.

Thus, there is a bijection:

$$\{\text{probability kernels}\} \longleftrightarrow \{\text{Markov generators}\}$$

We will make a lot of use of this. For example:

Cor: Let Q_1, Q_2 be prob. kernels over $(S, \mathcal{B})^2$.

Then $\exists!$ prob kernel Q s.t. $L_Q = L_{Q_1} L_{Q_2}$.

Cor: Let Q_1, Q_2 be prob. kernels over $(S, \mathcal{B})^2$.
Then $\exists!$ prob kernel Q s.t. $L_Q = L_{Q_1} L_{Q_2}$.

Pf. 0. $L_{Q_1} L_{Q_2}$ is a composition of linear operators on $\mathbb{B}(S, \mathcal{B})$, $\therefore$ is a linear operator on $(S, \mathcal{B})$.

1. $L_{Q_1} L_{Q_2}(1) =$
2. If $f \geq 0$, $\quad L_{Q_2}(f) \geq 0$
3. $f_n \to f$ boundedly $\Rightarrow \quad L_{Q_2}(f_n) \to L_{Q_2}(f) \quad$ boundedly.

We can write down the Q explicitly:

$$Q(x, B) = L_{Q_1} L_{Q_2}(\mathbb{1}_B)(x)$$

Special Case:

If S is countable (and $\mathcal{B} = 2^S$), then

$$Q_k(x, B) = \sum_{y \in B} Q_k(x, y)$$

and so $\quad (LQ_1 LQ_2 f)(x) =$

Prop: Let $Q_1, \dots, Q_n : (S, \mathcal{B})^2 \to [0,1]$ be prob. kernels, and let $\nu \in \mathrm{Prob}(S, \mathcal{B})$. Then $\exists!\ \mu \in \mathrm{Prob}(S^{n+1}, \mathcal{B}^{\otimes n+1})$ s.t.

$$\int_{S^{n+1}} f \, d\mu = \int_S \nu(dx_0) \int_S Q_1(x_0, dx_1) \int_S Q_2(x_1, dx_2) \cdots \int_S Q_n(x_{n-1}, dx_n) \, f(x_0, x_1, \dots, x_n)$$

for $f \in \mathbb{B}(S^{n+1}, \mathcal{B}^{\otimes n+1})$.

Pf. Begin by restricting to $f = f_0 \otimes f_1 \otimes \cdots \otimes f_n$

$\therefore \int_{S^{n+1}} f \, d\mu =$