Given a probability kernel over $(S_1,\mathcal{B}_1), (S_2,\mathcal{B}_2)$

$$Q : S_1 \times \mathcal{B}_2 \to [0,1]$$

we can integrate bounded measurable functions:

$$(L_Q f)(x) = \int f(y)\, Q(x,dy), \qquad f \in \mathbb{B}(S_2,\mathcal{B}_2).$$

and the result is a new measurable function of $x \in S_1$.
This function is also bounded:

$$\left| \int f(y)\, Q(x,dy) \right| \leq \int |f(y)|\, |Q(x,dy)| \leq \sup_y |f(y)| \cancelto{1}{\int Q(x,dy)} . \quad \forall x \in S_1$$

Def: Equip $\mathbb{B}(S_2,\mathcal{B}_2)$ with the norm $\|f\|_\infty := \sup |f|$.
For $f \in \mathbb{B}(S_2,\mathcal{B}_2)$, define $L_Q f \in \mathbb{B}(S_1,\mathcal{B}_1)$ by

Thus $L_Q : \mathbb{B}(S_2,\mathcal{B}_2) \to \mathbb{B}(S_1,\mathcal{B}_1)$. We call this operator a **Markov generator**.

$$\|L_Q f\|_\infty \leq \|f\|_\infty .$$

Prop: Given a probability kernel Q over $(S_1, \mathcal{B}_1), (S_2, \mathcal{B}_2)$ the associated Markov generator

$$L_Q : \mathbb{B}(S_2, \mathcal{B}_2) \to \mathbb{B}(S_1, \mathcal{B}_1)$$

has the following properties.

0. L_Q is a linear operator. ✓
1. $L_Q(1) = 1$ ✓
2. $L_Q(f) \geq 0$ if $f \geq 0$. ✓
3. If $f_n \in \mathbb{B}(S_2, \mathcal{B}_2)$ and $f_n \to f$ boundedly, then $L_Q f_n \to L_Q f$ boundedly. ✓

Pf. 0. Linearity of $\int$

1. $\int 1 Q(x, dy) = 1$.
2. $L_Q f(x) = \int f(y) Q(x, dy) \geq 0$.
3. $f_n \to f$ boundedly means $f_n \to f$ pointwise & $\|f_n\|_\infty \leq M < \infty \ \forall n$.

$\|L_Q f_n\|_\infty \leq \|f_n\|_\infty \leq M$.

$$(L_Q f_n)(x) = \int f_n(y) Q(x, dy)$$

$$\xrightarrow{DCT} \int \overset{\downarrow pt}{f(y)}\, Q(x, dy) = L_Q f(x)$$

Note: If $L : \mathbb{B}(S_2, \mathcal{B}_2) \to \mathbb{B}(S_1, \mathcal{B}_1)$ satisfies

0. L is linear, and
2. $Lf \geq 0$ if $f \geq 0$, then it follows that

$$f \leq g \Rightarrow Lf \leq Lg$$

$$0 = L(0) \leq L(\underset{\geq 0}{g - f}) = Lg - Lf$$

$$\therefore |Lf| \leq L|f|$$

$$\pm f \leq |f| \quad \therefore \pm L(f) = L(\pm f) \leq L|f|$$

And if 1. $L(1) = 1$ holds, then $f \leq \|f\|_\infty$

$$\Rightarrow |Lf|(x) \leq (L|f|)(x) \leq L(\|f\|_\infty) = \|f\|_\infty L(1) = \|f\|_\infty. \quad \forall x.$$

$$\therefore \|Lf\|_\infty \leq \|f\|_\infty.$$

If $f \in \mathbb{B}_{\mathbb{C}}(S_2, \mathcal{B}_2)$, we still have $|Lf| \leq L(|f|)$:

$$(Lf)(x) = |Lf(x)| e^{i\theta_x}$$

$$\therefore |Lf(x)| = (Lf)(x) e^{-i\theta_x} = L(e^{-i\theta_x} f)(x)$$

$$= L(Re(e^{-i\theta_x} f))(x) + i \cancel{L(Im(e^{-i\theta_x} f))(x)} \to 0$$

$$Re(e^{-i\theta_x} f)(x) \leq |e^{-i\theta_x} f|(x) = |f(x)|$$

Prop: Let $L : \mathbb{B}(S,\mathcal{B}) \to \mathbb{B}(S,\mathcal{B})$ be a linear operator satisfying 1-3:

1. $L(\mathbb{1}) = \mathbb{1}$
2. $L(f) \geq 0$ if $f \geq 0$.
3. If $f_n \to f$ boundedly in $\mathbb{B}(S,\mathcal{B})$, then $L(f_n) \to L(f)$ boundedly in $\mathbb{B}(S,\mathcal{B})$.

Then $Q(x,B) := L(\mathbb{1}_B)(x)$ defines a probability kernel over $(S,\mathcal{B})^2$, and $L_Q = L$.

Pf. As noted on the last slide, 1-2 $\Rightarrow$ $Lf \leq Lg$ when $f \leq g$.

$0 \leq \mathbb{1}_B \leq 1 \Rightarrow \quad 0 = L(0) \leq L(\mathbb{1}_B) \leq L(1) = \mathbb{1}$.

$B = \bigsqcup_{k=1}^{\infty} B_k \Rightarrow \mathbb{1}_B = \sum_{k=1}^{\infty} \mathbb{1}_{B_k}$. Set $f_n = \sum_{k=1}^{n} \mathbb{1}_{B_k}$ $\quad f_n \to \mathbb{1}_B$ boundedly.

$$\therefore \quad L(f_n) \to L(\mathbb{1}_B) = Q(\cdot,B)$$

$$\sum_{k=1}^{n} L(\mathbb{1}_{B_k}) \quad \therefore \quad \lim_{n\to\infty} \sum_{k=1}^{n} Q(\cdot,B_k)$$

$\therefore \; Q(x,\cdot) \in \mathrm{Prob}(S,\mathcal{B})$. $\quad Q(\cdot,B) = L(\mathbb{1}_B) \in \mathbb{B}(S,\mathcal{B})$

Thus $Q(x,B) := L(\mathbb{1}_B)(x)$ defines a prob. kernel.

We've left to show that $L = L_Q$: i.e. want

$$L(f) = L_Q(f) = \int f(y)\, Q(\cdot, dy)$$

True for $f = \mathbb{1}_B$: $\int \mathbb{1}_B(y)\, Q(\cdot, dy) = Q(\cdot, B) = L(\mathbb{1}_B)$

Now use Dynkin's multiplicative systems theorem. ///

Thus, there is a bijection:

$$\{\text{probability kernels}\} \longleftrightarrow \{\text{Markov generators}\}$$

$$Q \longmapsto L(f) = \int f(y)\, Q(\cdot, dy)$$

$$Q(\cdot, B) = L(\mathbb{1}_B) \longleftarrow L$$

We will make a lot of use of this. For example:

Cor: Let Q_1, Q_2 be prob. kernels over $(S, \mathcal{B})^2$.
Then $\exists!$ prob kernel Q s.t. $L_Q = L_{Q_1} L_{Q_2}$.

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Then $\exists!$ prob kernel Q s.t. $L_Q = L_{Q_1} L_{Q_2}$.

Pf. 0. $L_{Q_1} L_{Q_2}$ is a composition of linear operators on $\mathbb{B}(S, \mathcal{B})$, $\therefore$ is a linear operator on $(S, \mathcal{B})$.

1. $L_{Q_1}(L_{Q_2}(\mathbb{1})) = L_{Q_1}(\mathbb{1}) = \mathbb{1}$.
2. If $f \geq 0$, $g = L_{Q_2}(f) \geq 0$ $\therefore L_{Q_1} L_{Q_2}(f) = L_{Q_1}(g) \geq 0$.
3. $f_n \to f$ boundedly $\Rightarrow$ $g_n = L_{Q_2}(f_n) \to L_{Q_2}(f) = g$ boundedly.

$$\therefore \underset{\displaystyle L_{Q_1} L_{Q_2}(f_n)}{\underset{\shortparallel}{L_{Q_1}(g_n)}} \to \underset{\displaystyle L_{Q_1} L_{Q_2}(f)}{\underset{\shortparallel}{L_{Q_1}(g)}} \text{ boundedly}$$

We can write down the Q explicitly:

$$Q(x, B) = L_{Q_1} \underbrace{L_{Q_2}(\mathbb{1}_B)}_{g}(x) = \int g(y) Q_1(x, dy) = \int Q_2(y, B) Q_1(x, dy)$$

$g(y) = Q_2(y, B)$ Generally $L_Q f(x) = \int Q_1(x, dy) \int Q_2(y, dz) f(z)$

Special Case:

If S is countable (and $\mathcal{B} = 2^S$), then

$$Q_k(x, B) = \sum_{y \in B} Q_k(x, y)$$

and so

$$(L_{Q_1} L_{Q_2} f)(x) = \sum_{y, z \in S} Q_1(x, y) Q_2(y, z) f(z)$$

$$= \sum_{z \in S} \Big(\underbrace{\sum_{y \in S} Q_1(x, y) Q_2(y, z)}_{\text{matrix mult.}} \Big) f(z)$$

$$L_Q f(x) = \sum_{y \in S} Q(x, y) f(y)$$

basis $\{\mathbb{1}_{\{x\}}\}_{x \in S}$

$\{Q(x, y)\}_{x, y \in S}$ is the matrix of the lin. trans. L_Q

Prop: Let $Q_1, \dots, Q_n : (S, \mathcal{B})^2 \to [0,1]$ be prob. kernels, and let $\nu \in \mathrm{Prob}(S, \mathcal{B})$. Then $\exists!\ \mu \in \mathrm{Prob}(S^{n+1}, \mathcal{B}^{\otimes n+1})$ s.t.

$$\int_{S^{n+1}} f \, d\mu = \int_S \nu(dx_0) \int_S Q_1(x_0, dx_1) \int_S Q_2(x_1, dx_2) \cdots \int_S Q_n(x_{n-1}, dx_n) \, f(x_0, x_1, \dots, x_n)$$

for $f \in \mathbb{B}(S^{n+1}, \mathcal{B}^{\otimes n+1})$.

Pf. Begin by restricting to $f = f_0 \otimes f_1 \otimes \cdots \otimes f_n$

$$\therefore \int_{S^{n+1}} f \, d\mu = \int f_0(x_0) \nu(dx_0) \int f_1(x_1) Q_1(x_0, dx_1) \cdots \underbrace{\int f_n(x_n) Q_n(x_{n-1}, dx_n)}_{(LQ_n f_n)(x_{n-1})}$$

$$= \int f_0 \, LQ_1 (f_1 (\cdots (LQ_{n-1} (f_{n-1} LQ_n f_n)) \cdots)) \, d\nu$$

Take $f_k = \mathbb{1}_{B_k}$, defines μ on $B_0 \times B_1 \times \cdots \times B_n$. ///