Conditional Expectation and Independence

Ey[X] is the "best guess" at X using only information in B. What if B has no

information about X?

Propi Let X: (2,7) > (SB) be a random variable,

and let BSJ be a sub-5-field.

If 5(X), & are independent, and f:S->IR is s.t. f(X) EL (2,F,P), then  $E_{\mathcal{H}}[f(X)] = E[f(X)] as.$ Conversely, if holds for all fe B(S, B), then G(X), & are independent.

 $Pf. \implies$  Let YEB(SJ). Then E[f(X)Y]

 $( \in ) \mathbb{E}[f(X)Y]$ 

Conditioning on a Random Variable / Vector

If X: (SF) > (SB) (think S= Rd),

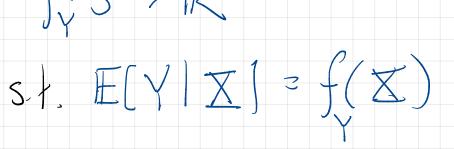
and YE L'(SJP), we denote

 $\pi E_{\sigma(X)}[Y] = E[Y|_{\sigma(X)}]$ 

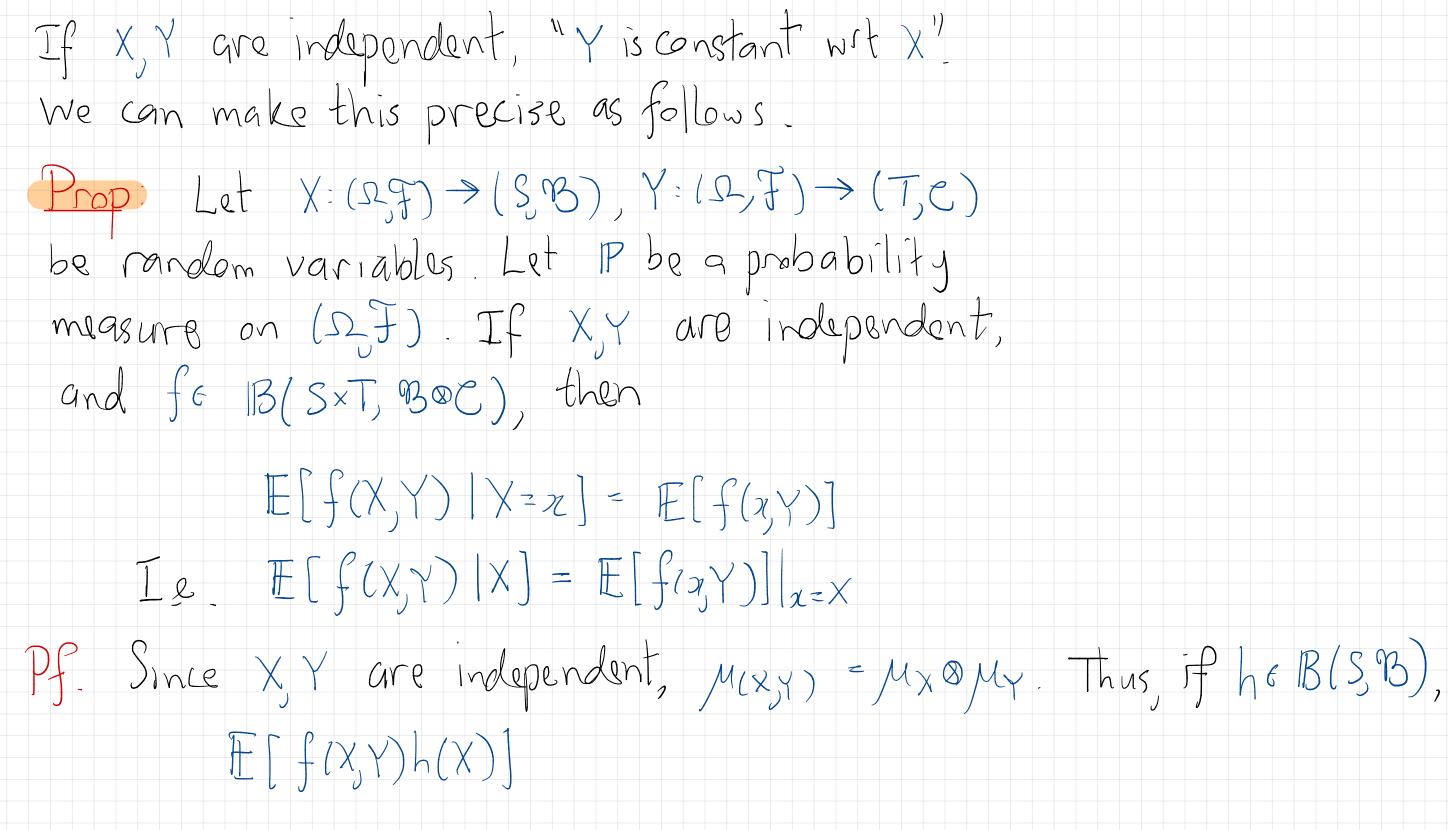
This is in  $L^{1}(\Omega, \sigma(\mathbf{X}), \mathbb{P})$ . In particular, it is  $\sigma(\mathbf{X})/9b(\mathbb{R})$ -measurable. By the Doob-Dynkin representation, there is a  $\frac{9}{3}/\frac{9}{3}(\mathbb{R})$ -measurable function  $f_{i}^{2}S \rightarrow \mathbb{R}$ 

Notation:  $f_{\gamma}(s) =$ :

Equivalently: fy:S>R is characterized by  $E[Y-h(X)] = E[f_Y(X)h(X)] \forall h \in B(S, B)$ 







Eq. Suppose (X,Y) has a joint density C= exy We want to identify E[f(X,Y)|X] This means we want, Yhe B(IR, B(IR)),  $\mathbb{E}[\mathbb{E}[f(X,Y)|X]h(X)] = \mathbb{E}[f(X,Y)h(X)]$ Note: since (X,Y) has a joint density, X has a  $P(X \in A) = P((X, y) \in A \times \mathbb{R}) = \int_{X \in A} p(x, y) dy dy = \int_{A} \left( \int_{\mathbb{R}} p(x, y) dy \right) dx$  $I_2 = e_X(x) = \int_B e(x,y) ely$ 

olensity

Prop. Let (X,Y) have density e=exy. Let ex (x) = fex, (x, y) dy be the marginal dansity of X. Define  $\frac{(x, y(x, y))}{(x)}$  $e_{YIX}(y|z) :=$ Then for  $f \in B(\mathbb{R}^2)$ ,  $\mathbb{E}[f(X,Y)|X] = g(X) \quad \text{where} \quad g(x) = \int f(x,y) e_{Y|X}(y|x) dy$ We defined g to make this true if (2xcx) > 0Claim: If (2xcx) = 0 then