Extending $\mathbb{E}_{\mathcal{G}}$ Beyond L^1

Prop: If $X \geq 0$ is $\mathcal{F}$-measurable, and $\mathcal{G} \subseteq \mathcal{F}$ is a sub-σ-field, $\exists !$ (up to null sets) $\mathcal{G}$-measurable r.v. $\tilde{X}$ satisfying

$$\bigstar \qquad \mathbb{E}[X : B] = \mathbb{E}[\tilde{X} : B] \quad \forall B \in \mathcal{G}.$$

(If $X \in L^1$, $\tilde{X} = \mathbb{E}_{\mathcal{G}}[X]$; $\therefore$ we call it $\mathbb{E}_{\mathcal{G}}[X]$ even if $X \notin L^1$.)

Pf. To define $\tilde{X}$, note that $X \wedge n \in L^1$ so $\mathbb{E}_{\mathcal{G}}[X \wedge n]$ exists, and by the monotonicity of $\mathbb{E}_{\mathcal{G}}$ on L^1, $\mathbb{E}_{\mathcal{G}}[X \wedge n] \uparrow$. $\therefore$ $\mathbb{E}_{\mathcal{G}}[X] := \lim_{n \to \infty} \mathbb{E}_{\mathcal{G}}[X \wedge n]$.

Now, by MCT: $\mathbb{E}[\mathbb{E}_{\mathcal{G}}[X] \mathbb{1}_B] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}_{\mathcal{G}}[X \wedge n] \mathbb{1}_B] = \lim_{n \to \infty} \mathbb{E}[(X \wedge n) \mathbb{1}_B]$
$= \mathbb{E}[X \mathbb{1}_B]$.

$\bigstar$ uniquely specifies $\tilde{X}$ (up to null sets) same proof as before. ///

Note: $\mathbb{E}_{\mathcal{G}}$ is monotone: if $0 \leq X \leq Y$, a.s.,

$$\mathbb{E}_{\mathcal{G}}[X] = \lim_{n\to\infty} \mathbb{E}_{\mathcal{G}}[X \wedge n] \leq \lim_{n\to\infty} \mathbb{E}_{\mathcal{G}}[Y \wedge n] = \mathbb{E}_{\mathcal{G}}[Y].$$

Theorem: $\mathbb{E}_{\mathcal{G}}$ satisfies the standard integral convergence results:

(cMCT) If $0 \le X_n \le X_{n+1}$ a.s., then $\lim_{n\to\infty} \mathbb{E}_{\mathcal{G}}[X_n] = \mathbb{E}_{\mathcal{G}}[\lim_{n\to\infty} X_n]$ a.s.

(cFatou) If $X_n \ge 0$ a.s., then $\mathbb{E}_{\mathcal{G}}[\liminf_{n\to\infty} X_n] \le \liminf_{n\to\infty} \mathbb{E}_{\mathcal{G}}[X_n]$ a.s.

(cDCT) If $X_n \to X$ a.s. and $|X_n| \le Y \in L^1$ a.s., then $\mathbb{E}_{\mathcal{G}}[X_n] \to \mathbb{E}_{\mathcal{G}}[X]$ a.s. and in L^1.

Pf. (cMCT) By monotonicity of $\mathbb{E}_{\mathcal{G}}$, $\mathbb{E}_{\mathcal{G}}[X_n] \uparrow$.

Fix $B \in \mathcal{G}$. Then by the standard MCT:

$$\mathbb{E}\left[\lim_{n\to\infty} \mathbb{E}_{\mathcal{G}}[X_n]\, \mathbb{1}_B\right] = \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}_{\mathcal{G}}[X_n]\mathbb{1}_B\right] = \lim_{n\to\infty} \mathbb{E}[X_n \mathbb{1}_B]$$
$$= \mathbb{E}\left[\lim_{n\to\infty} X_n \mathbb{1}_B\right] = \mathbb{E}\left[\mathbb{E}_{\mathcal{G}}\left[\lim_{n\to\infty} X_n\right]\mathbb{1}_B\right]$$

(cFatou) Let $Y_k = \inf_{n\ge k} X_n$. Then $Y_k \le X_k$ $\forall k$ and $Y_k \uparrow$. So by (cMCT),

$$\mathbb{E}_{\mathcal{G}}\left[\liminf_{n\to\infty} X_n\right] = \mathbb{E}_{\mathcal{G}}\left[\lim_{k\to\infty} Y_k\right] = \liminf_{k\to\infty} \mathbb{E}_{\mathcal{G}}[Y_k] \le \liminf_{k\to\infty} \mathbb{E}_{\mathcal{G}}[X_k]$$

(cDCT) First: by usual DCT, $X_n \to X$ in L^1. Thus

$$\|E_{\mathcal{G}}[X_n] - E_{\mathcal{G}}[X]\|_{L^1} = \|E_{\mathcal{G}}[X_n - X]\|_{L^1} \leq \|X_n - X\|_{L^1} \longrightarrow 0.$$

$$\therefore\ E_{\mathcal{G}}[X_n] \longrightarrow E_{\mathcal{G}}[X] \quad \text{in } L^1.$$

Now, since $0 \leq Y \pm X_n$, by (cFatou),

$$E_{\mathcal{G}}[Y] \pm E_{\mathcal{G}}[X] = E_{\mathcal{G}}[Y \pm X] = E_{\mathcal{G}}\left[\liminf_{n\to\infty}(Y \pm X_n)\right] \leq \liminf_{n\to\infty} E_{\mathcal{G}}[Y \pm X_n] = \begin{cases} E_{\mathcal{G}}[Y] + \liminf\limits_{n\to\infty} E_{\mathcal{G}}[X_n] & + \\ E_{\mathcal{G}}[Y] - \limsup\limits_{n\to\infty} E_{\mathcal{G}}[X_n] & - \end{cases}$$

$$\limsup_{n\to\infty} E_{\mathcal{G}}[X_n] \underset{=}{\leq} E_{\mathcal{G}}[X] \underset{=}{\leq} \liminf_{n\to\infty} E_{\mathcal{G}}[X_n] \leq$$

$$\therefore\ E_{\mathcal{G}}[X] = \lim_{n\to\infty} E_{\mathcal{G}}[X_n]$$

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Theorem: (Conditional Jensen's Inequality)

Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subseteq \mathcal{F}$ is a sub-σ-field.

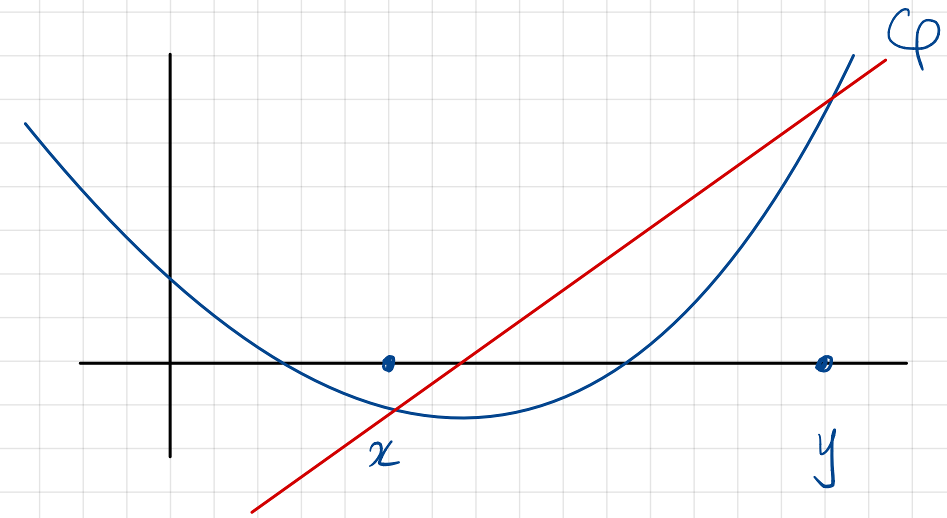
If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex, and $\varphi(X) \in L^1$, then

$\varphi(\mathbb{E}_{\mathcal{G}}[X]) \in L^1(\Omega, \mathcal{G}, \mathbb{P})$, and

$$\varphi(\mathbb{E}_{\mathcal{G}}[X]) \leq \mathbb{E}_{\mathcal{G}}[\varphi(X)] \text{ a.s.} \quad \therefore \mathbb{E}[\varphi(\mathbb{E}_{\mathcal{G}}[X])] \leq \mathbb{E}[\varphi(X)].$$

Pf. To simplify our lives, we'll also impose the stronger assumption $\varphi \in C^2$ and $\varphi'' > 0$,

$$\therefore \varphi'(x) \leq \frac{\varphi(y) - \varphi(x)}{y - x} \quad \forall x < y$$



See [Driver, 10.38] and [Driver, 17.68] in general.

$$\therefore \varphi(y) \geq \varphi(x) + \varphi'(x)(y - x) \quad \forall x, y$$

$$\therefore \varphi(X) \geq \varphi(x) + \varphi'(x)(X - x) \text{ a.s.}$$

$$\mathbb{E}_{\mathcal{G}}[\varphi(X)] \geq \varphi(x) + \varphi'(x)(\mathbb{E}_{\mathcal{G}}[X] - x) \text{ a.s.}, \quad \forall x \in \mathbb{Q}.$$

$$\hookrightarrow \geq \sup_{x \in \mathbb{Q}} [\varphi(x) + \varphi'(x)(\mathbb{E}_{\mathcal{G}}[X] - x)] \text{ a.s.}$$

Claim: $\sup_{x\in\mathbb{Q}} [\varphi(x)+\varphi'(x)(y-x)] = \varphi(y)$.

$= \underbrace{\sup_{x\in\mathbb{R}} (\quad " \quad)}$ by continuity of φ, φ'.

achieved @ critical pt. $0 = \frac{d}{dx}(\varphi(x)+\varphi'(x)(y-x))$

$= \cancel{\varphi'(x)} + \varphi''(x)(y-x) + \cancel{\varphi'(x)(-1)}$

$\therefore \mathbb{E}_{\mathcal{G}}[\varphi(X)] \geq \sup_{x\in\mathbb{Q}} [\varphi(x)+\varphi'(x)(\mathbb{E}_{\mathcal{G}}[X]-x)] = \varphi(\mathbb{E}_{\mathcal{G}}[X])$. ✓

Now $\varphi(X)\in L^1$, $\therefore \mathbb{E}_{\mathcal{G}}[\varphi(X)]\in L^1$.

Going back: $\varphi(y) \geq \varphi(x)+\varphi'(x)(y-x) \quad \forall x,y$

$\therefore \varphi(\mathbb{E}_{\mathcal{G}}[X]) \geq \varphi(x)+\varphi'(x)(\mathbb{E}_{\mathcal{G}}[X]-x)$

Thus $|\varphi(\mathbb{E}_{\mathcal{G}}[X])| \leq |\mathbb{E}_{\mathcal{G}}[\varphi(X)]| \vee |\varphi(x)+\varphi'(x)(\mathbb{E}_{\mathcal{G}}[X]-x)| \quad \forall x$.

$\therefore$? ? ?

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Cor: For $1 \le p < \infty$, $\mathbb{E}_{\mathcal{Y}} : L^p \to L^p$ is a contraction.

Pf. $\varphi_p(x) = |x|^p$ is convex. Thus, by (cJensen),
if $\varphi_p(X) \in L^1$ $\quad (|X|^p \in L^1$ i.e. $X \in L^p)$

$$\varphi_p(\mathbb{E}_{\mathcal{Y}}[X]) \le \mathbb{E}_{\mathcal{Y}}[\varphi_p(X)]$$

$$\mathbb{E}[|\mathbb{E}_{\mathcal{Y}}[X]|^p] \le \mathbb{E}[\mathbb{E}_{\mathcal{Y}}[|X|^p]] = \mathbb{E}[|X|^p].$$

$$\therefore \|\mathbb{E}_{\mathcal{Y}}[X]\|_p \le \|X\|_p.$$

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