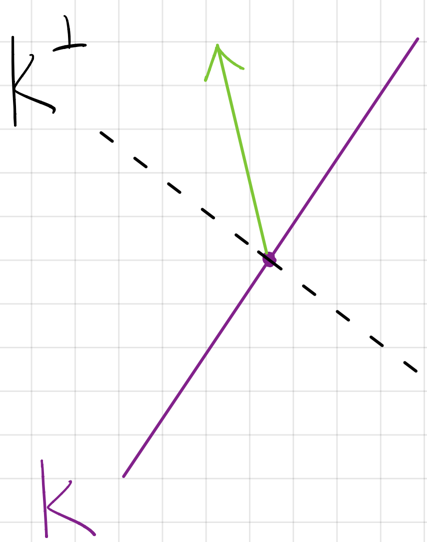
Orthogonal Projection

If H is a finite dimensional inner product space, and $K \subseteq H$ is any subspace, there is an **orthogonal projection**

$$P_K : H \twoheadrightarrow K$$

with the properties that



$$P_K(v) = v \quad \forall v \in K$$

$$P_K(w) = 0 \quad \text{if} \quad w \in K^\top$$

If we can find an orthonormal basis $\{e_n\}$ for K, then

$$P_K(v) =$$

We will use this same idea in the Hilbert space $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Hilbert Spaces

A Hilbert space is a complete inner product space.

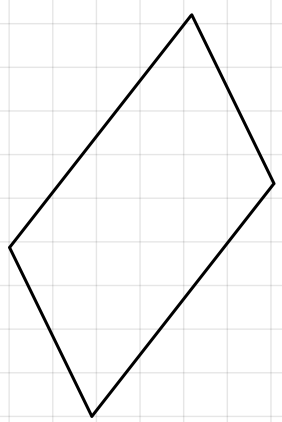
Eg. $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $\langle X, Y \rangle :=$

In any inner product space, we have Pythagoras' Thm:

if $X \perp Y$, $\|X+Y\|^2$

Also, we have the Parallelogram Law:

$\|X+Y\|^2 + \|X-Y\|^2$



If $K \subseteq H$ is a linear subspace, it is also an inner product space (in the same inner product).

K is a Hilbert space iff

Prop: If $K \subseteq H$ is a subspace, and $X \in H$, there is a unique closest element $Y \in K$ to X:

$$\|X - Y\|^2 = d(X, K)^2 := \inf_{Z \in K} \|X - Z\|^2.$$

Pf. For any $Y, Z \in K$, $\|Y - Z\|^2 = \|(Y-X) - (Z-X)\|^2$

$$= 2(\|Y-X\|^2 + \|Z-X\|^2) - \|Y-X+Z-X\|^2$$

$$\therefore \|Y-Z\|^2 + 4\left\|\frac{Y+Z}{2} - X\right\|^2 = 2(\|Y-X\|^2 + \|Z-X\|^2)$$

Thus $\|Y-Z\|^2 + 4\,d(X,K)^2 \leq 2(\|Y-X\|^2 + \|Z-X\|^2)$

Uniqueness: If $d(X,K)^2 = \|X-Y\|^2 = \|X-Z\|^2$,

Existence: Let $Y_n \in K$ with $\|X - Y_n\|^2 \leq d(X,K)^2 + \frac{1}{n}$.

$$\therefore \|Y_n - Y_m\|^2 + 4\,d(X,K)^2 \leq 2(\|Y_n - X\|^2 + \|Y_m - X\|^2)$$

Prop: The unique closest point $Y \in K$ to X is also the unique element $Y \in K$ satisfying

$$X - Y \perp K$$

Pf. If Y is the closest point, for any $Z \in K$, consider

$$\mathbb{R} \ni t \mapsto \alpha(t) = \| X - (Y + tZ) \|^2$$

by assumption, $\alpha(0) = \min \alpha$

Conversely, if $Y \in K$ with $X - Y \perp K$, then for any $Z \in K$,

$\| X - Z \|^2$

Theorem: Given a Hilbert space H and a closed subspace $K \subseteq H$, there is a unique linear transformation $P_K : H \twoheadrightarrow K$ s.t.

- P_K is Lip_1-continuous.
- $P_K(Y) = \quad \forall Y \in K$
- $P_K(Z) = 0 \quad \forall Z \in K^\perp$
- $\langle P_K(X), Y \rangle = \langle X, P_K(Y) \rangle \quad \forall X, Y \in H$

Moreover, if $L \subseteq K$ is another closed subspace, then $P_K P_L = P_L P_K = P_L$.

The transformation P_K, the orthogonal projection onto K, can be defined by $P_K(X) =$ the unique element in K closest to X.

Pf. We've shown that there is a unique closest point $Y = P_K(X)$ to K, and it is characterized by $(P_K(X) - X) \perp K$.

If $X_1, X_2 \in H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, and $Y \in K$,

$$\langle (\alpha_1 P_K(X_1) + \alpha_2 P_K(X_2)) - (\alpha_1 X_1 + \alpha_2 X_2), Y \rangle =$$

If $X \in H$, $X = X - P_K(X) + P_K(X)$

$$\langle P_K(X), Y \rangle = \langle P_K(X), Y \rangle + \langle P_K(X), P_K(Y) - Y \rangle$$

Finally, if $L \subseteq K \subseteq H$, if $X \in H$, $P_L(X) \in L \subseteq K \therefore P_K P_L(X)$

For the reverse, for $X, Y \in H$,

$$\langle P_L P_K(X), Y \rangle = \langle P_K(X), P_L(Y) \rangle = \langle X, P_K P_L(Y) \rangle$$