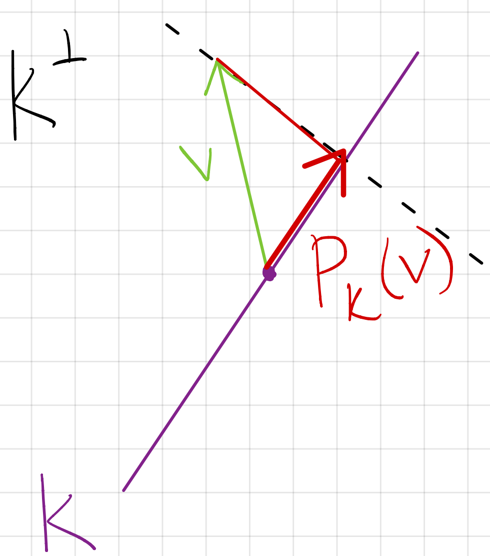
Orthogonal Projection

If H is a finite dimensional inner product space, and $K \subseteq H$ is any subspace, there is an orthogonal projection

$$P_K : H \twoheadrightarrow K$$

with the properties that



$$P_K(v) = v \qquad \forall v \in K$$

$$P_K(w) = 0 \quad \text{if} \quad w \in K^\top \leftarrow \langle v, w \rangle = 0 \ \forall v \in K.$$

If we can find an orthonormal basis $\{e_n\}$ for K, then

$$P_K(v) = \sum_n \left\langle v, \frac{e_n}{\|e_n\|} \right\rangle \frac{e_n}{\|e_n\|}$$

We will use this same idea in the Hilbert space $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Hilbert Spaces

A **Hilbert space** is a complete inner product space.

Eg. $H = L^2(\Omega, \mathcal{F}, \mathbb{P})$ with $\langle X, Y \rangle := \mathbb{E}[XY]$

In any inner product space, we have **Pythagoras' Thm**:

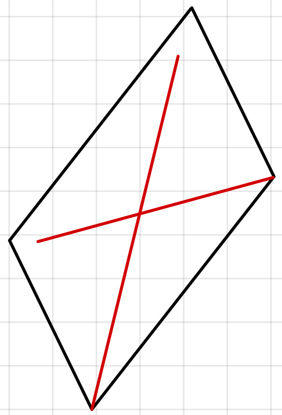
if $X \perp Y$, $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$

$\langle X, Y \rangle = 0$ $\quad \langle X+Y, X+Y \rangle = \langle X, X \rangle + \underbrace{\cancel{\langle X, Y \rangle}}_{0} + \underbrace{\cancel{\langle Y, X \rangle}}_{0} + \langle Y, Y \rangle$

Also, we have the **Parallelogram Law**:

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\left(\|X\|^2 + \|Y\|^2\right)$$

$$\langle X+Y, X+Y \rangle + \langle X-Y, X-Y \rangle = \langle X, X \rangle + \cancel{\langle X, Y \rangle} + \cancel{\langle Y, X \rangle} + \langle Y, Y \rangle + \langle X, X \rangle - \cancel{\langle X, Y \rangle} - \cancel{\langle Y, X \rangle} + \langle Y, Y \rangle$$



If $K \subseteq H$ is a linear subspace, it is also an inner product space (in the same inner product).

K is a Hilbert space iff K is <u>closed</u> in H.

Prop: If $K \subseteq H$ is a closed subspace, and $X \in H$, there is a unique closest element $Y \in K$ to X:

$$\|X - Y\|^2 = d(X,K)^2 := \inf_{Z \in K} \|X - Z\|^2.$$

Pf. For any $Y, Z \in K$, $\|Y-Z\|^2 = \|(Y-X)-(Z-X)\|^2$

parallelogram law $\rightarrow$ $= 2(\|Y-X\|^2 + \|Z-X\|^2) - \underbrace{\|Y-X+Z-X\|^2}_{\|Y+Z-2X\|^2 = 4\|\frac{Y+Z}{2}-X\|^2}$

$$\therefore \|Y-Z\|^2 + 4\|\underbrace{\tfrac{Y+Z}{2}}_{\in K} - X\|^2 = 2(\|Y-X\|^2 + \|Z-X\|^2)$$

$$\|\tfrac{Y+Z}{2} - X\|^2 \geq d(X,K)^2$$

Thus $\|Y-Z\|^2 + 4\,d(X,K)^2 \leq 2(\|Y-X\|^2 + \|Z-X\|^2)$

<u>Uniqueness</u>: If $d(X,K)^2 = \|X-Y\|^2 = \|X-Z\|^2$, $\therefore \|Y-Z\|^2 \leq 0 \Rightarrow Y = Z$.

<u>Existence</u>: Let $Y_n \in K$ with $\|X - Y_n\|^2 \leq d(X,K)^2 + \frac{1}{n}$.

$$\therefore \|Y_n - Y_m\|^2 + \cancel{4\,d(X,K)^2} \leq 2(\|Y_n - X\|^2 + \|Y_m - X\|^2) \leq \cancel{4\,d(X,K)^2} + 2(\tfrac{1}{n} + \tfrac{1}{m})$$

$\therefore \underbrace{\{Y_n\}}_{\in K \,\leftarrow\, \text{closed}}$ is Cauchy in H. $\therefore Y_n \to Y \in H$. $\therefore Y \in K$.

$$\|X - Y\| \leq \underbrace{\|X - Y_n\|}_{\leq d(X,K) + \frac{1}{n}} + \underbrace{\|Y_n - Y\|}_{\to 0}$$

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Prop: The unique closest point $Y \in K$ to X is also the unique element $Y \in K$ satisfying

$$X - Y \perp K \quad \text{i.e. } \langle X-Y, Z\rangle = 0 \ \forall Z \in K.$$

Pf. If Y is the closest point, for any $Z \in K$, consider

$$\mathbb{R} \ni t \mapsto \alpha(t) = \|X - (Y + tZ)\|^2 = \|X-Y\|^2 - 2t\langle X-Y, Z\rangle + \|Z\|^2 t^2$$

by assumption, $\alpha(0) = \min \alpha$ $\qquad 0 \overset{!}{=} \alpha'(0) = -2\langle X-Y, Z\rangle.$

Conversely, if $Y \in K$ with $X - Y \perp K$, then for any $Z \in K$,

$$\|X-Z\|^2 = \|X - Y + \underbrace{Y - Z}_{\in K}\|^2 \underset{\text{Pythagoras}}{=} \|X-Y\|^2 + \underbrace{\|Y-Z\|^2}_{\geq 0} \geq \|X-Y\|^2$$

$\therefore$ Y = unique minimizer of $d(X, K)$.

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Theorem: Given a Hilbert space H and a closed subspace $K \subseteq H$, there is a unique linear transformation $P_K : H \twoheadrightarrow K$ s.t.

- P_K is Lip_1-continuous.
- $P_K(Y) = Y \quad \forall Y \in K$ ← clear.
- $P_K(Z) = 0 \quad \forall Z \in K^\perp$ $\quad 0 - Z \in K^\perp$
- $\langle P_K(X), Y \rangle = \langle X, P_K(Y) \rangle \quad \forall X, Y \in H$

Moreover, if $L \subseteq K$ is another closed subspace, then $P_K P_L = P_L P_K = P_L$.
The transformation P_K, the orthogonal projection onto K, can be defined by $P_K(X) =$ the unique element in K closest to X.

Pf. We've shown that there is a unique closest point $Y = P_K(X)$ to K, and it is characterized by $(P_K(X) - X) \perp K$.
If $X_1, X_2 \in H$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, and $Y \in K$, $\quad P_K(\alpha_1 X_1 + \alpha_2 X_2)$

$$\langle (\alpha_1 P_K(X_1)) + \alpha_2 P_K(X_2)) - (\alpha_1 X_1 + \alpha_2 X_2), Y \rangle = \alpha_1 \underbrace{\langle P_K(X_1) - X_1, Y \rangle}_{0} + \alpha_2 \underbrace{\langle P_K(X_2) - X_2, Y \rangle}_{0} = 0$$

If $X \in H$, $X = \underbrace{X - P_K(X)}_{\perp K} + \underbrace{P_K(X)}_{\in K}$

$$\therefore \|X\|^2 = \|X - P_K(X)\|^2 + \|P_K(X)\|^2 \geq \|P_K(X)\|^2$$

$$\therefore \|P_K(X)\| \leq \|X\| \quad \forall X \in H$$

$$\therefore \|P_K(X) - P_K(Y)\| = \|P_K(X-Y)\| \leq \|X-Y\| \quad \therefore P_K \in Lip_1.$$

$$\langle P_K(X), Y\rangle = \langle P_K(X), Y\rangle + \langle \underbrace{P_K(X)}_{\in K}, \underbrace{P_K(Y) - Y}_{\in K^\perp}\rangle$$

$$= \langle P_K(X), P_K(Y)\rangle = \langle X, P_K(Y)\rangle.$$

Finally, if $L \subseteq K \subseteq H$, if $X \in H$, $P_L(X) \in L \subseteq K \quad \therefore P_K(\underbrace{P_L(X)}_{\in K}) = P_L(X)$

For the reverse, for $X, Y \in H$,

$$\langle P_L P_K(X), Y\rangle = \langle P_K(X), P_L(Y)\rangle = \langle X, P_K P_L(Y)\rangle = \langle X, P_L(Y)\rangle$$

$$= \langle P_L(X), Y\rangle.$$

$$0 = \langle P_L P_K(X) - P_L(X), Y\rangle \quad Y = P_L P_K(X) - P_L(X)$$