Given $(\Omega, \mathcal{F}, \mathbb{P})$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$,
we have the conditioned probability measure

$$\mathbb{P}_A(B) = \mathbb{P}[B \mid A] = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

We can then take integrals of random variables $X \in L^1(\Omega, \mathcal{F}, \mathbb{P}_A)$

$$\mathbb{E}_A[X] = \mathbb{E}[X \mid A] = \int X \, d\mathbb{P}_A.$$

Lemma: $L^1(\mathbb{P}) = L^1(\mathbb{P}_A)$, and $\mathbb{E}_A[X] = \dfrac{\mathbb{E}[X : A]}{\mathbb{P}(A)}$. $\mathbb{E}[X:A] \leftarrow \mathbb{E}[X \mathbb{1}_A]$

Pf. If X is simple, $X = \sum_j \alpha_j \mathbb{1}_{B_j}$, then

$$\int X \, d\mathbb{P}_A = \sum_j \alpha_j \mathbb{P}_A(B_j) = \sum_j \alpha_j \frac{\mathbb{P}(A \cap B_j)}{\mathbb{P}(A)} = \frac{1}{\mathbb{P}(A)} \int \underbrace{\sum_j \alpha_j \mathbb{1}_{B_j \cap A}}_{\underbrace{\left(\sum_j \alpha_j \mathbb{1}_{B_j}\right) \mathbb{1}_A}_{X}} d\mathbb{P} = \frac{\mathbb{E}[X \mathbb{1}_A]}{\mathbb{P}(A)}.$$

For general $X = X^+ - X^-$ $\qquad X_n^+ + X_n^- \leq X^+ + X^- = |X| \in L^1$

$\mathbb{E}[X:A]/\mathbb{P}(A)$ $\qquad 0 \leq X_n^{\pm} \underset{\to}{\leq} X^{\pm}$ $\qquad \overset{\shortparallel}{|X_n^+ - X_n^-|}$

$$\overbrace{\left(\int X_n^+ \mathbb{1}_A \, d\mathbb{P} - \int X_n^- \mathbb{1}_A \, d\mathbb{P}\right)}^{\uparrow} \Big/ \mathbb{P}(A) = \int (X_n^+ - X_n^-) \, d\mathbb{P}_A \to \int X \, d\mathbb{P}_A \quad ///$$

If $\{A_n\}_{n=1}^{\infty}$ is a partition of Ω by events with $\mathbb{P}(A_n) > 0$, we can string the different $\mathbb{E}_{A_n}$ functionals together into a single function:

$$\mathbb{E}_{\{A_n\}_{n=1}^{\infty}}[X] = \sum_{n=1}^{\infty} \mathbb{E}_{A_n}[X] \mathbb{1}_{A_n}$$

I.e., $\omega \mapsto \begin{cases} \vdots \\ \mathbb{E}_{A_n}[X] = \dfrac{E[X : A_n]}{\mathbb{P}(A_n)} & \text{for } \omega \in A_n . \\ \vdots \end{cases}$

Labeling: instead of indexing this by the partition $\{A_n\}_{n=1}^{\infty}$, it is more convenient (and suggestive) to label it by $\mathcal{G} = \sigma(\{A_n\}_{n=1}^{\infty})$.

This makes sense because:

Exercise: If $A_n \in \mathcal{F}$ with $\Omega = \bigsqcup_{n=1}^{\infty} A_n$, then

$$\sigma(\{A_n\}_{n=1}^{\infty}) = \{ \bigsqcup_{n \in \Lambda} A_n : \Lambda \subseteq \mathbb{N} \}$$

Thus, we can "recover" the partition from $\mathcal{G}$ (minimal generating set)

Cor: If $\mathcal{G} = \sigma(\{A_n\}_{n=1}^{\infty})$ for some partition $\{A_n\}_{n=1}^{\infty}$, then $E_{\mathcal{G}}[X]$ is $\mathcal{G}$-measurable.

Pf. $E_{\mathcal{G}}[X] = \sum_{n=1}^{\infty} E_{A_n}[X] \mathbb{1}_{A_n}$ is a linear combination of the indicator functions $\mathbb{1}_{A_n} \in \mathcal{G}$. ///

$$\sigma(E_{\mathcal{G}}[X]) = \mathcal{G}.$$

Note: even in this special case where $\mathcal{G}$ is generated by a countable partition, it is not practically possible to recover the partition from $\mathcal{G}$ (unless $\mathcal{G}$, and $\therefore$ the partition, is finite)

(Motivating) Eg. Suppose $Y : \Omega \to S$ is a random variable whose state space S is countable, and $\mathbb{P}(Y=s) > 0$ for each $s \in S$. Then $\{Y^{-1}(s) : s \in S\}$ forms a partition of Ω.

$$\{Y^{-1}(s) : s \in S\} \quad \sigma\{Y^{-1}(s) : s \in S\} = \sigma(Y)$$

For $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mathbb{E}[X|Y] = \mathbb{E}[X|\sigma(Y)] = \sum_{s \in S} \mathbb{E}_{\{Y=s\}}[X] \mathbb{1}_{\{Y=s\}} = \sum_{s \in S} \frac{\mathbb{E}[X : Y=s]}{\mathbb{P}(Y=s)} \mathbb{1}_{\{Y=s\}}$$

$$\mathbb{E}[X|Y](\omega) = \frac{\mathbb{E}[X \mathbb{1}_{Y=Y(\omega)}]}{\mathbb{P}(Y=Y(\omega))}$$

only depends on ω through $Y(\omega)$

$\mathbb{E}[X|Y]$ is $\sigma(Y)$-measurable

$\therefore$ by Doob-Dynkin, $\mathbb{E}[X|Y] = g(Y)$ for some $g : S \to \mathbb{R}$

$$g(s) = \text{"}\mathbb{E}[X|Y=s]\text{"}$$

$$= \mathbb{E}_{P_{\{Y=s\}}}[X]$$

There is a more "invariant" way to define conditional expectation, owing to the following.

Prop: Let $\{A_n\}_{n=1}^{\infty}$ be a partition of Ω by positive probability events, and set $\mathcal{G} = \sigma(\{A_n\}_{n=1}^{\infty})$.
If $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})$, then

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}_{\mathcal{G}}[X]\, Y].$$

Pf. Step 1: the case that Y is simple. So $Y = \sum_j \beta_j \mathbb{1}_{B_j} \leftarrow B_j \in \mathcal{G}$

$B_j = \bigsqcup_{k \in \Lambda_j} A_k$

$\therefore \quad \mathbb{E}[XY] = \sum_{k=1}^{\infty} \beta_k' \mathbb{E}[X \mathbb{1}_{A_k}]$

$\mathbb{E}_{\mathcal{G}}[X] = \sum_{j=1}^{\infty} \mathbb{E}_{A_j}[X] \mathbb{1}_{A_j} \longleftarrow Y = \sum_{k=1}^{\infty} \beta_k' \mathbb{1}_{A_k}$

So $\mathbb{E}[\mathbb{E}_{\mathcal{G}}[X] Y] = \sum_{j,k=1}^{\infty} \mathbb{E}_{A_j}[X] \beta_k' \mathbb{E}[\mathbb{1}_{A_j} \mathbb{1}_{A_k}] \longleftarrow \delta_{jk} \mathbb{E}[\mathbb{1}_{A_k}] = \delta_{jk} \mathbb{P}(A_k)$

$$= \sum_{k=1}^{\infty} \mathbb{E}_{A_k}[X] \underbrace{\beta_k' \mathbb{P}(A_k)}_{= \beta_k' \mathbb{E}[X \mathbb{1}_{A_k}]}$$

Step 2: If $Y \in L^2(\Omega, \mathcal{G}, \mathbb{P}) \subseteq L^1(\Omega, \mathcal{G}, \mathbb{P})$,

$\exists$ sequence of simple Y_n ($\mathcal{G}$-measurable) s.t.

$Y_n \to Y$ a.s. and $|Y_n| \leq |Y| \in L^2$

$\therefore$ $XY_n \to XY$ a.s. and $|XY_n| \leq |XY| \in L^1$ b/c $X, Y \in L^2$ by C-S

$\therefore$ By DCT, $\mathbb{E}[XY_n] \to \mathbb{E}[XY]$ $\qquad |\mathbb{E}_{\mathcal{G}}[X]Y_n| \leq |\mathbb{E}_{\mathcal{G}}[X]Y| \in L^1$

$$\begin{array}{ccc} \mathbb{E}[XY_n] & \longrightarrow & \mathbb{E}[XY] \\ \| & & \| \\ \mathbb{E}[\mathbb{E}_{\mathcal{G}}[X]Y_n] & \xrightarrow{\checkmark} & \mathbb{E}[\mathbb{E}_{\mathcal{G}}[X]Y] \end{array}$$

DCT

Note: $\|\mathbb{E}_{\mathcal{G}}[X]\|_{L^2}^2 = \mathbb{E}\left[\left(\sum_{j=1}^{n} \mathbb{E}_{A_j}[X]\mathbb{1}_{A_j}\right)^2\right] = \sum_{j=1}^{\infty} \underbrace{\mathbb{E}_{A_j}[X]^2}\, \mathbb{P}(A_j) \leq \sum_{j=1}^{\infty} \mathbb{E}[X^2\mathbb{1}_{A_j}]$

$= \mathbb{E}[X^2]$

$= \|X\|_{L^2}^2$

$(\mathbb{E}[X\mathbb{1}_{A_j} \cdot \mathbb{1}_{A_j}])^2$

$\leq \mathbb{E}[(X\mathbb{1}_{A_j})^2]\,\mathbb{E}[\mathbb{1}_{A_j}^2]$

$= \mathbb{E}[X^2\mathbb{1}_{A_j}]\,\mathbb{P}(A_j)$

$$\longrightarrow \left(\frac{\mathbb{E}[X\mathbb{1}_{A_j}]}{\mathbb{P}(A_j)}\right)^2 \leq \frac{\mathbb{E}[X^2\mathbb{1}_{A_j}]}{\mathbb{P}(A_j)}$$

///