Conditional Probability

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
and $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$, we can form
the new probability measure $\mathbb{P}_B : \mathcal{F} \to [0,1]$

$$\mathbb{P}[A \mid B] = \mathbb{P}_B(A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad A \cap B = AB.$$

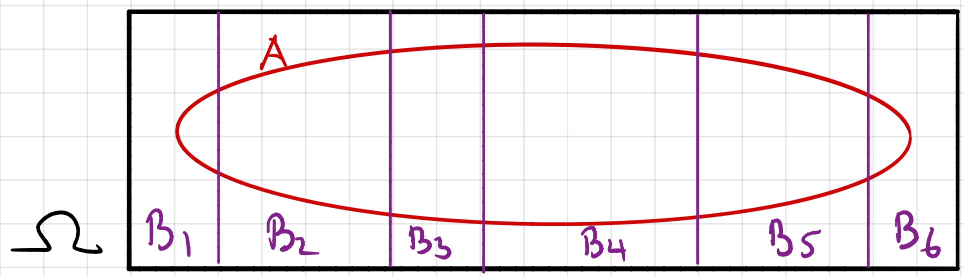
Intuition: we have observed event a has occurred;
how does that affect the "posterior probability"
of other events?

prior $\to \mathbb{P}(A)$ $\qquad$ $\mathbb{P}_B(A) \leftarrow$ posterior.

We can combine different conditional measures $\mathbb{P}_{B_j}$ well,
especially if the events B_j form a partition of Ω.

Law of Total Probability

If $B_1, B_2, \ldots, B_n$ partition Ω (disjoint, $B_1 \cup \cdots \cup B_n = \Omega$, $\mathbb{P}(B_j) > 0$)



then for any event A:

$$\mathbb{P}(A) = \mathbb{P}(AB_1 \cup AB_2 \cup \cdots \cup AB_n) = \sum_{j=1}^{n} \mathbb{P}(B_j)\,\mathbb{P}(A \mid B_j).$$

Eg. 90% of coins are fair, 9% are biased to come up heads 60%.
1% are biased to come up heads 80%.

You find a coin on the street. How likely is it to come up heads?

$B_1 = \{\text{fair coins}\}$ $\quad \mathbb{P}(B_1) = 90\%$ $\quad \mathbb{P}(A \mid B_1) = 50\%$

$B_2 = \{60\%\ \text{heads}\}$ $\quad \mathbb{P}(B_2) = 9\%$ $\quad \mathbb{P}(A \mid B_2) = 60\%$

$B_3 = \{80\%\ \text{heads}\}$ $\quad \mathbb{P}(B_3) = 1\%$ $\quad \mathbb{P}(A \mid B_3) = 80\%$

$A = \{\text{heads}\}$

$\longrightarrow \mathbb{P}(A) = 51.2\%$

2.2

Question:

90% of coins are fair. 9% are biased to come up heads 60%.
1% are biased to come up heads 80%.

You find a coin on the street. You toss it, and it comes up heads.

How likely is it that this coin is heavily biased?

$$\mathbb{P}(B_3 \mid H) \neq \mathbb{P}(H \mid B_3) = 80\%$$

Eg. According to Forbes Magazine, as of April 10, 2019, there are
2208 billionaires in the world.
1964 of them are men.

$$\mathbb{P}(M \mid B) = \frac{1964}{2208} \doteq 89\% \neq \mathbb{P}(B \mid M)$$

Bayes' Rule

(A relationship between $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$)

Let $B_1, B_2, \dots, B_n$ partition the sample space. Then for any event A with $\mathbb{P}(A) > 0$,

$$\mathbb{P}(B_k|A) = \mathbb{P}(B_k A) / \mathbb{P}(A)$$

$$= \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A|B_k)\mathbb{P}(B_k)}{\sum_{j=1}^{n} \mathbb{P}(A|B_j)\mathbb{P}(B_j)}$$

Eg. (Coins) $\mathbb{P}(C_{80}|H)$

$$= \frac{\mathbb{P}(C_{80}H)}{\mathbb{P}(H)} = \frac{\mathbb{P}(H|C_{80})\mathbb{P}(C_{80})}{\mathbb{P}(H)}$$

$$\frac{(80\%)(1\%)}{51.2\%} = \frac{\mathbb{P}(H|C_{80})\mathbb{P}(C_{80})}{\mathbb{P}(H|C_{50})\mathbb{P}(C_{50}) + \mathbb{P}(H|C_{60})\mathbb{P}(C_{60}) + \mathbb{P}(H|C_{80}) \cdot \mathbb{P}(C_{80})}$$

$\approx 1.56\%$

Epidemiological Confusion

An HIV test is 99% accurate. (1% false positives, 1% false negatives)
0.33% of US residents have HIV.

If you test positive, what is the probability you have HIV?

(a) 99%

(b) 1%

(c) 25%

(d) 0.33%

(e) There is not enough information to answer.

$T = \{\text{positive test}\}$

$H = \{\text{have HIV}\}$

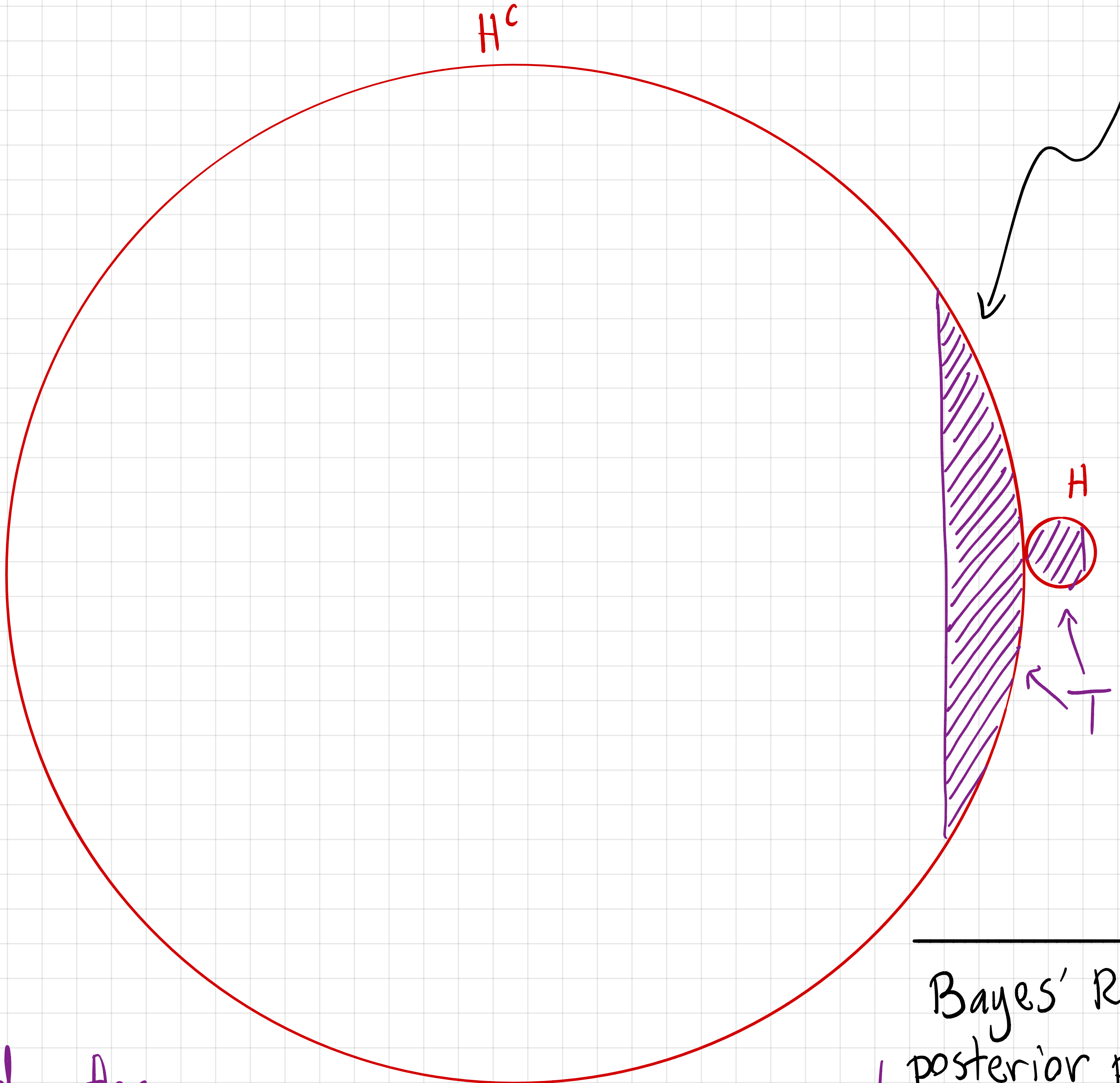
$\Omega = H \cup H^c$

$\mathbb{P}(T \mid H^c) = 1\% = \mathbb{P}(T^c \mid H) = 1 - \mathbb{P}(T \mid H)$

$\mathbb{P}(H) = 0.33\%$

$$\mathbb{P}(H \mid T) = \frac{\mathbb{P}(HT)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T \mid H)\,\mathbb{P}(H)}{\mathbb{P}(T \mid H)\,\mathbb{P}(H) + \mathbb{P}(T \mid H^c)\,\mathbb{P}(H^c)}$$

$$= \frac{(0.99)(0.0033)}{(0.99)(0.0033) + (0.01)(99.67\%)} \doteq 24.69\%$$



even though this part of T is only 1% of H^c, it is **3 times as big** as the part of T in H (which takes up 99% of H).

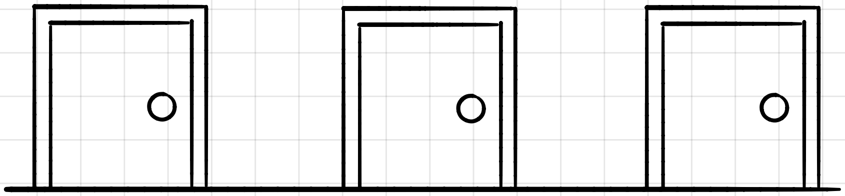
This is possible because H^c **dwarfs** H in this example.

Bayes' Rule shows that posterior probabilities are **highly sensitive** to prior inputs.

Redo the Calc w/ $\mathbb{P}(H) = 0.03$

$\Rightarrow \mathbb{P}(H|T) = 75.3\%$ | $\mathbb{P}(H) = 0.3$, $\mathbb{P}(H|T) = 97.7\%$

The Monty Hall Problem



At the climax of a gameshow, you are shown 3 doors. The host tells you that, behind one of them is a valuable prize (a car), while the other two hide nothing of value (goats).

You choose one. The host then opens one of the two doors you did not choose, revealing a goat. He then asks you if you want to stick with your original choice, or switch to the other closed door.

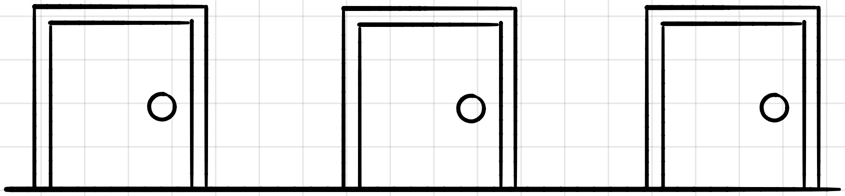
Should you switch??

(a) Yes.

(b) No.

(c) Doesn't matter.

The Monty Hall Problem



Let's decide to call the door you chose originlly #1.

$\therefore$ Monty will open #2 or #3. We'll focus our analysis on #2.

$B_i = \{ \text{the car is behind door } \#i \}$.

$A = \{ \text{Monty opens door } \#2 \}$

We want to know $\mathbb{P}(B_3 | A)$.

$\mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(B_3) = \frac{1}{3}$

$\mathbb{P}(A | B_2) = 0$

$\mathbb{P}(A | B_3) = 1$

$\mathbb{P}(A | B_1) = \frac{1}{2}$

$$\mathbb{P}(B_3 | A) = \frac{\mathbb{P}(B_3 A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_3)\mathbb{P}(A|B_3)}{\mathbb{P}(B_1)\mathbb{P}(A|B_1) + \mathbb{P}(B_2)\mathbb{P}(A|B_2) + \mathbb{P}(B_3)\mathbb{P}(A|B_3)}$$

$$= \frac{(1/3)\cdot 1}{(1/3)1 + 0 + (1/3)(1/2)} = \frac{2}{3}.$$