Quantitative CLT via Stein's Method

We've seen that, for any random variable W,
if $Z \overset{d}{=} \mathcal{N}(0,1)$, then

$$d_{W_1}(W,Z) \le \sup_{\mathcal{F}} \left| \mathbb{E}[f'(W) - Wf(W)] \right|$$

where $\mathcal{F} = \{ f : \mathbb{R} \to \mathbb{R} : \sup|f| \le 2, \ \sup|f'| \le \sqrt{\tfrac{2}{\pi}}, \ \sup|f''| \le 2 \}$

We will apply this to $W = \frac{S_n}{\sqrt{n}}$ where

$$S_n = X_1 + \cdots + X_n, \quad \{X_n\}_{n=1}^{\infty} \text{ iid } \mathbb{E}[X_j] = 0, \ \mathbb{E}[X_j^2] = 1, \ \mathbb{E}[|X_j|^3] < \infty.$$

$$\mathbb{E}[Wf(W)] = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathbb{E}[X_j f(W)]$$

Key idea: $W_j := W - \frac{1}{\sqrt{n}} X_j = \frac{1}{\sqrt{n}} \sum_{i \ne j} X_i$ $\quad \therefore W_j, X_j$ independent

$$\therefore \mathbb{E}[X_j f(W_j)] = \underbrace{\mathbb{E}[X_j]}_{0} \mathbb{E}[f(W_j)] = 0.$$

$$\mathbb{E}[X_j f(W)] = \mathbb{E}[X_j (f(W) - f(W_j))]$$

Now, putting on Calculus hats: $f(w) - f(x) \approx f'(x)(w-x)$

More precisely: if f is twice differentiable @ x, then: $\quad f(w) - f(x) = (w-x)f'(x) + \frac{1}{2}(w-x)^2 f''(\xi)$

$$= \mathbb{E}\left[X_j \left(f(W) - f(W_j) - (W - W_j) f'(W_j) + (W - W_j) f'(W_j)\right)\right]$$

$$= \mathbb{E}\Big[X_j \cdot \frac{1}{2} \underbrace{(W - W_j)^2}_{=\left(\frac{1}{\sqrt{n}} X_j\right)^2} f''(\xi_j)\Big] + \mathbb{E}\Big[X_j \underbrace{(W - W_j)}_{\frac{1}{\sqrt{n}} X_j} f'(W_j)\Big]$$

$$= \frac{1}{2n} \mathbb{E}[X_j^3 f''(\xi_j)] \qquad \frac{1}{\sqrt{n}} \mathbb{E}[X_j^2 f'(W_j)]$$

$$= \frac{1}{\sqrt{n}} \mathbb{E}[\underbrace{\cancel{X_j^2}}_{1}]\, \mathbb{E}[f'(W_j)]$$

(1) $\mathbb{E}[Wf(W)] = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbb{E}[X_j f(W)] = \frac{1}{2n^{3/2}}\sum_{j=1}^{n}\mathbb{E}[X_j^3 f''(\xi_j)] + \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[f'(W_j)]$

for some rv's ξ_j.

Now, $f'(W_j) - f'(W) = f''(\eta_j)\underbrace{(W_j - W)}_{= -\frac{1}{\sqrt{n}}X_j}$ for some rv η_j

$$\therefore \left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[f'(W_j)]\right) - \mathbb{E}[f'(W)]$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left(\mathbb{E}[f'(W_j) - f'(W)]\right) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}\left[f''(\eta_j)\left(-\frac{1}{\sqrt{n}}X_j\right)\right]$$

(2) $\mathbb{E}[f'(W)] = \left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[f'(W_j)]\right) + \frac{1}{n^{3/2}}\sum_{j=1}^{n}\mathbb{E}[f''(\eta_j)X_j]$

Combining (1) & (2):

$$\mathbb{E}[Wf(W) - f'(W)] = \frac{1}{2n^{3/2}}\sum_{j=1}^{n}\mathbb{E}[X_j^3 f''(\xi_j)] - \frac{1}{n^{3/2}}\sum_{j=1}^{n}\mathbb{E}[f''(\eta_j)X_j]$$

$$\therefore \left|\mathbb{E}[Wf(W) - f'(W)]\right| \leq \frac{1}{2n^{3/2}}\sup|f''|\sum_{j=1}^{n}\mathbb{E}[|X_j|^3] + \frac{1}{n^{3/2}}\sup|f''|\sum_{j=1}^{n}\mathbb{E}[|X_j|]$$

Now (and only now) using the fact that the X_j are identically distributed, we have

$$\frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[|X_j|^3] = \mathbb{E}[|X_1|^3] \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}[|X_j|] = \mathbb{E}[|X_1|] \leq \mathbb{E}[|X_1|^3]$$

Hölder's inequality

$$\mathbb{E}[|X| \cdot 1] \leq \mathbb{E}[|X_1|^3]^{1/3}\, \mathbb{E}[1^{3/2}]^{2/3}$$

$$\therefore |\mathbb{E}[Wf(W) - f'(W)]|$$

$$\leq \frac{1}{2\sqrt{n}} \sup|f''|\, \mathbb{E}[|X_1|^3] + \frac{1}{\sqrt{n}}\sup|f''|\, \mathbb{E}[|X_1|] .$$

$(\mathbb{E}[X_1^2] = 1 \Rightarrow \mathbb{E}[|X_1|^3] \geq 1$

$\leq \mathbb{E}[|X_1|^3]$

Theorem: If $\{X_n\}_{n\geq 1}^{\infty}$ are iid L^3 random variables with $\mathbb{E}[X_j]=0$, $\mathbb{E}[X_j^2]=1$, and $S_n = X_1 + \cdots + X_n$, then if $Z \overset{d}{=} \mathcal{N}(0,1)$,

$$\sup_{f\in\mathcal{F}} |\mathbb{E}[Wf(W) - f'(W)]| = d_{W_1}\left(\frac{S_n}{\sqrt{n}}, Z\right) \leq \frac{\mathbb{E}[|X_1|^3] + 2\mathbb{E}[|X_1|]}{\sqrt{n}} \leq \frac{3\,\mathbb{E}[|X_1|^3]}{\sqrt{n}} .$$

Pf. $\quad f\in\mathcal{F} \Rightarrow \sup|f''| \leq 2$.

Cor: $\quad d_{Kol}\left(\frac{S_n}{\sqrt{n}}, Z\right) \leq \frac{\text{Const.}}{n^{1/4}}$.