Gaussian Integration by Parts

In a previous HW you showed that if $Z \stackrel{d}{=} \mathcal{N}(0,1)$,

$$\mathbb{E}[Z f(Z)] = \mathbb{E}[f'(Z)]$$

for all $f \in C^1$ s.t. $Zf(Z)$, $f(Z)$, and $f'(Z)$ are in L^1.

Actually don't need $f \in C^1$; just need f' to exist a.e. and for $\int_0^t f'(x)\,dx = f(t)$ for a.e. t. [This happens iff h is absolutely continuous.]

The converse is true!

Stein's Lemma:

Let W be a random variable s.t. $\mathbb{E}[f'(W) - Wf(W)] = 0$ for all absolutely continuous functions f with $\sup|f'| < \infty$. Then $W \stackrel{d}{=} \mathcal{N}(0,1)$.

Lemma 1: Let $\Phi(t) = F_Z(t) = \mathbb{P}(Z \le t)$. Then

$$\forall t>0, \qquad 1-\Phi(t) \le \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2\pi}\,t}\right\} e^{-t^2/2}$$

Pf.

$$1-\Phi(t) = \mathbb{P}(Z>t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$\le \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}\,t}\left(-e^{-x^2/2}\right\Big|_{x=t}^{x=\infty}$$

Lemma 2: For each $t \in \mathbb{R}$, the ODE

$$f_t'(w) - w f_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$$

has a unique bounded solution f_t, given by

$$f_t(w) = e^{w^2/2} \int_w^\infty e^{-x^2/2}\left(\Phi(t) - \mathbb{1}_{(-\infty,t]}(x)\right) dx,$$

and $\sup|f_t| \le \sqrt{\frac{2}{\pi}}$, $\sup|f_t'| \le 2$.

Pf.

$$f_t'(w) - w f_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$$

General solution:

$$e^{-w^2/2} f_t(w) = -\int_w^\infty e^{-x^2/2}\left(\mathbb{1}_{(-\infty,t]}(x) - \Phi(t)\right) dx + C$$

I.e. $$f_t(w) = e^{w^2/2}\int_w^\infty e^{-x^2/2}\left(\mathbb{1}_{(-\infty,t]}(x) - \Phi(t)\right) dx + C e^{w^2/2}$$

$$f_t(w) = \sqrt{2\pi}\, e^{w^2/2} \begin{cases} \Phi(w)(1-\Phi(t)) & w \le t \\ \Phi(t)(1-\Phi(w)) & w \ge t \end{cases}$$

$$f_t'(w) = w f_t(w) + \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$$

Stein's Lemma:

Let W be a random variable s.t. $\mathbb{E}[f'(W) - Wf(W)] = 0$ for all absolutely continuous functions f with $\sup|f'| < \infty$.
Then $W \overset{d}{=} \mathcal{N}(0,1)$.

Pf. Apply to $f = f_t$ from Lemma 2, which satisfies $\sup|f_t'| \leq 2$.

$$f_t'(w) - wf_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$$

$$\therefore \quad 0 = \mathbb{E}[f_t'(W) - Wf_t(W)]$$

Better yet:

Cor: If W is any random variable, and f_t is as above, then

$$|F_W(t) - \Phi(t)| = |\mathbb{E}[f_t'(W) - Wf_t(W)]|$$

It is possible, with a careful analysis, to use this

$$d_{Kol}(W, Z) = \sup_{t \in \mathbb{R}} |\mathbb{E}[f_t'(W) - W f_t(W)]|$$

↑

where $f_t'(w) - w f_t(w) = \mathbb{1}_{(-\infty, t]}(w) - \Phi(t)$

$$f_t(w) = \sqrt{2\pi}\, e^{w^2/2} \begin{cases} \Phi(w)(1-\Phi(t)) & w \le t \\ \Phi(t)(1-\Phi(w)) & w \ge t \end{cases}$$

to prove the Berry-Esseen theorem: with $W = \frac{S_n}{\sqrt{n}}$ for $S_n = \sum_{j=1}^n X_j$, $\{X_j\}$ iid, $\mathbb{E}[X_j] = 0$, $\mathbb{E}[X_j^2] = 1$

$$\forall t \in \mathbb{R}. \quad |\mathbb{E}[f_t'(W) - W f_t(W)]| \le \frac{3}{\sqrt{n}} \mathbb{E}[|X_1|^3].$$

It is a bit annoying.

It would be better if we worked with a "test function" h that is smooth, so that the associated f_h is even smoother...

Stein's Method

Instead of trying to bound $d_{Kol}(W, Z)$,
bound $d_{W_1}(W, Z) = \sup_{\|h\|_{Lip} \leq 1} \left| \int h \, d\mu_W - \int h \, d\mu_Z \right|$

Following the intuition from Stein's Lemma, we want to bound this by

Consider the ODE

$$f_h'(w) - w f_h(w) = h(w) - \Phi(h)$$

Stein's Bounds: Let $h \in Lip(\mathbb{R})$. The ODE

$$f_h'(w) - w f_h(w) = h(w) - \Phi(h)$$

has a unique bounded solution, given by

$$f_h(w) = e^{w^2/2} \int_w^\infty e^{-x^2/2} (\Phi(h) - h(x))\, dx$$

and f_h is C^1, and its derivative f_h' is differentiable. Moreover

$$\sup|f_h| \leq 2\, \|h\|_{Lip}$$

$$\sup|f_h'| \leq \sqrt{\tfrac{2}{\pi}}\, \|h\|_{Lip}$$

$$\sup|f_h''| \leq 2\, \|h\|_{Lip}$$

See [CGS, Lemma 2.4]

Cor: For any random variable W, if $Z \overset{d}{=} \mathcal{N}(0,1)$, then

$$d_{W_1}(W, Z) \leq \sup\{ |\mathbb{E}[f'(W) - Wf(W)]| : f \in \mathcal{H}\}$$

where $\mathcal{H} = \{ f: \mathbb{R} \to \mathbb{R} : \sup|f| \leq 2,\ \sup|f'| \leq \sqrt{\tfrac{2}{\pi}},\ \sup|f''| \leq 2\}$