Gaussian Integration by Parts

In a previous HW you showed that if $Z \overset{d}{=} \mathcal{N}(0,1)$,

$$\mathbb{E}[Z f(Z)] = \mathbb{E}[f'(Z)]$$

for all $f \in C^1$ s.t. $\cancel{Zf(Z)}$, $\cancel{f(Z)}$, and $f'(Z)$ are in L^1.

Actually don't need $f \in \cancel{C^1}$; just need f' to exist a.e. and for $\int_{-\infty}^{t} f'(x)\,dx = f(t)$ for a.e. t. [This happens iff f is absolutely continuous.]

The converse is true!

Stein's Lemma:

Let W be a random variable s.t. $\mathbb{E}[f'(W) - Wf(W)] = 0$ for all absolutely continuous functions f with $\sup|f'| < \infty$. Then $W \overset{d}{=} \mathcal{N}(0,1)$.

Lemma 1: Let $\Phi(t) = F_Z(t) = \mathbb{P}(Z \le t)$. Then

$$\forall t>0,\ \Phi(-t) = 1-\Phi(t) \le \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2\pi}\,t}\right\} e^{-t^2/2}$$

Pf. $\Phi(0) = \frac{1}{2}$, $\Phi \uparrow$

$$1-\Phi(t) = \mathbb{P}(Z>t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

For $x \ge t$, $\frac{x}{t} \ge 1$

$$\le \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}\,t}\left(-e^{-x^2/2}\right)\Big|_{x=t}^{x=\infty}$$

Lemma 2: For each $t \in \mathbb{R}$, the ODE

$$f_t'(w) - w f_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$$

$f_t(w) = f_{-t}(-w)$

has a unique bounded solution f_t, given by

Suffice to show for $w>0$

$$f_t(w) = e^{w^2/2}\int_w^\infty e^{-x^2/2}\left(\Phi(t) - \mathbb{1}_{(-\infty,t]}(x)\right)dx,$$

and $\sup|f_t| \le \sqrt{\frac{2}{\pi}}$, $\sup|f_t'| \le 2$.

Pf. $e^{-w^2/2}\left(f_t'(w) - w f_t(w)\right) = \left(\mathbb{1}_{(-\infty,t]}(w) - \Phi(t)\right) e^{-w^2/2}$

$$\overset{\parallel}{\frac{d}{dw}\left[e^{-w^2/2} f_t(w)\right]}$$

General solution:

$$e^{-w^2/2} f_t(w) = -\int_w^\infty e^{-x^2/2}\left(\mathbb{1}_{(-\infty,t]}(x) - \Phi(t)\right) dx + C$$

I.e. $$f_t(w) = e^{w^2/2}\underbrace{\int_w^\infty e^{-x^2/2}\Big(\underbrace{\mathbb{1}_{(-\infty,t]}(x) - \Phi(t)}_{|\cdot| \leq 1}\Big) dx}_{\leq \sqrt{2\pi}\,(1-\Phi(w)) \;\leq\; e^{-w^2/2}} + \cancel{C} e^{w^2/2}$$

$C = 0$.

$$f_t(w) = \sqrt{2\pi}\, e^{w^2/2} \begin{cases} \Phi(w)(1-\Phi(t)) & w \leq t \\ \Phi(t)(1-\Phi(w)) & w \geq t \end{cases}$$

$\leq \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2\pi}\, w} e^{-w^2/2}\right\} \leq \sqrt{\frac{2}{\pi}} e^{-w^2/2}$

$\frac{1}{\sqrt{2\pi}} \leq \frac{1}{2}$ if $w > \sqrt{\frac{2}{\pi}}$

$$|f_t'(w)| = \left|w f_t(w) + \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)\right| \leq 2.$$

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Stein's Lemma:

Let W be a random variable s.t. $\mathbb{E}[f'(W) - Wf(W)] = 0$ for all absolutely continuous functions f with $\sup|f'| < \infty$.

Then $W \stackrel{d}{=} \mathcal{N}(0,1)$.

Pf. Apply to $f = f_t$ from Lemma 2, which satisfies $\sup|f_t'| \leq 2$.

$$f_t'(w) - wf_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$$

$$\therefore \quad 0 = \mathbb{E}[f_t'(W) - Wf_t(W)] = \mathbb{E}[\mathbb{1}_{(-\infty,t]}(W) - \Phi(t)]$$

$$= \underbrace{\mathbb{P}(W \leq t)}_{F_W(t)} - \Phi(t) \qquad ///$$

Better yet:

Cor: If W is any random variable, and f_t is as above, then

$$|F_W(t) - \Phi(t)| = |\mathbb{E}[f_t'(W) - Wf_t(W)]|$$

$$\therefore \quad d_{Kol}(W,Z) = \sup_{t \in \mathbb{R}} |\mathbb{E}[f_t'(W) - Wf_t(W)]|$$

It is possible, with a careful analysis, to use this

$$d_{Kol}(W, Z) = \sup_{t \in \mathbb{R}} |\mathbb{E}[f_t'(W) - W f_t(W)]|$$

$$\uparrow$$

where $f_t'(w) - w f_t(w) = \mathbb{1}_{(-\infty,t]}(w) - \Phi(t)$

$\Phi(t) = \mathbb{P}(Z \le t)$
$= \mathbb{E}[\mathbb{1}_{(-\infty,t]}(Z)]$

$$f_t(w) = \sqrt{2\pi}\, e^{w^2/2} \begin{cases} \Phi(w)(1-\Phi(t)) & w \le t \\ \Phi(t)(1-\Phi(w)) & w \ge t \end{cases}$$

to prove the Berry-Esseen theorem: with $W = \frac{S_n}{\sqrt{n}}$ for $S_n = \sum_{j=1}^{n} X_j$, $\{X_j\}$ iid

$\mathbb{E}[X_j] = 0$ $\mathbb{E}[X_j^2] = 1$

$$\forall t \in \mathbb{R}. \quad |\mathbb{E}[f_t'(W) - W f_t(W)]| \le \frac{3}{\sqrt{n}} \mathbb{E}[|X_1|^3].$$

It is a bit annoying.

It would be better if we worked with a "test function" $h = \mathbb{1}_{(-\infty,t]}$ that is smooth, so that the associated f_h is even smoother...

Stein's Method

Instead of trying to bound $d_{Kol}(W, Z)$,
bound $d_{W_1}(W, Z) = \sup_{\|h\|_{Lip} \leq 1} \Big| \underbrace{\int h \, d\mu_W - \int h \, d\mu_Z}_{\mathbb{E}[h(W) - h(Z)]} \Big|$

Following the intuition from Stein's Lemma, we want to bound this by
$= \mathbb{E}[f'(W) - Wf(W)]$ for some $f = f_h$.

Consider the ODE

$$f_h'(w) - w f_h(w) = h(w) - \Phi(h)$$

$\Phi(h) = \mathbb{E}[h(Z)]$
$= \int h \, d\mathcal{N}(0,1)$

$$\mathbb{E}[f_h'(W) - W f_h(W)] = \mathbb{E}[h(W) - \Phi(h)]$$
$$= \mathbb{E}[h(W) - h(Z)]$$

$$\therefore \; d_{W_1}(W, Z) = \sup_{\|h\|_{Lip} \leq 1} \left| \mathbb{E}[f_h'(W) - W f_h(W)] \right|$$

Stein's Bounds: Let $h \in \mathrm{Lip}(\mathbb{R})$. The ODE

$$f_h'(w) - w f_h(w) = h(w) - \Phi(h)$$

has a unique bounded solution, given by

$$f_h(w) = e^{w^2/2} \int_w^\infty e^{-x^2/2} \left(\Phi(h) - h(x)\right) dx$$

and f_h is C^1, and its derivative f_h' is differentiable. Moreover

$$\left.\begin{aligned} \sup|f_h| &\leq 2\|h\|_{Lip} \\ \sup|f_h'| &\leq \sqrt{\tfrac{2}{\pi}}\|h\|_{Lip} \\ \sup|f_h''| &\leq 2\|h\|_{Lip} \end{aligned}\right\}$$

Similar analysis to Lemma 2, using Gaussian tail bounds

See [CGS, Lemma 2.4]

Cor: For any random variable W, if $Z \overset{d}{=} N(0,1)$, then

$$d_{W_1}(W, Z) \leq \sup\left\{ \left|\mathbb{E}[f'(W) - Wf(W)]\right| : f \in \mathcal{H} \right\}$$

where $\mathcal{H} = \left\{ f : \mathbb{R} \to \mathbb{R} : \sup|f| \leq 2,\ \sup|f'| \leq \sqrt{\tfrac{2}{\pi}},\ \sup|f''| \leq 2 \right\}$