Probability Metrics

We've seen the total variation metric on $\mathrm{Prob}(S,\mathcal{B})$

$$d_{TV}(\mu,\nu) = \sup_{B\in\mathcal{B}} |\mu(B)-\nu(B)|$$

This is an example of a (dual) probability metric:

$$d(\mu,\nu) = \sup_{h\in\mathcal{H}} \left|\int h\,d\mu - h\,d\nu\right|$$

↳ some class of functions on S.

Always a pseudo-metric; genuine metric if $\mathcal{H}$ is sufficiently rich.

Eg. Kolmogorov metric: $d_{Kol}(\mu,\nu) = \sup_{t\in\mathbb{R}} |F_\mu(t) - F_\nu(t)|$
(on $\mathrm{Prob}(\mathbb{R},\mathcal{B}(\mathbb{R}))$)

Wasserstein L^1 Distance

On $\mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, define

$$d_{W_1}(\mu, \nu) = \sup_{\|f\|_{Lip} \leq 1} \left| \int f \, d\mu - \int f \, d\nu \right|$$

i.e. $\mathcal{H} = \mathrm{Lip}_1(\mathbb{R})$

That doesn't look like what we called the Wasserstein metric in [Lecture 21.2]. But it actually is.

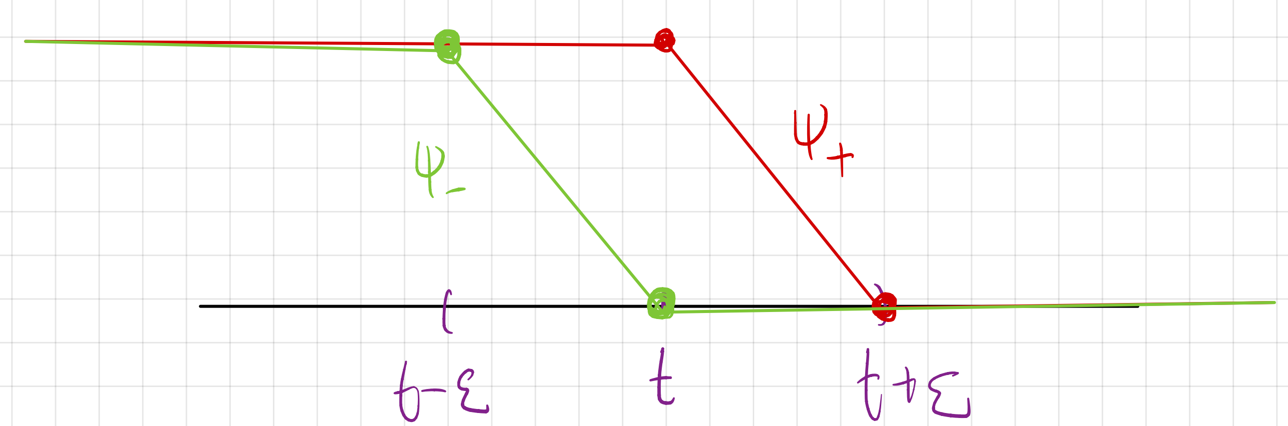
Prop: $$d_{W_1}(\mu, \nu) = \sup_{\|f\|_{Lip} \leq 1} \left| \int f \, d\mu - \int f \, d\nu \right|$$
$$= \int_{-\infty}^{\infty} |F_\mu(t) - F_\nu(t)| \, dt$$
$$= \inf \{ \mathbb{E}[|X-Y|] : (X, Y) \text{ is a coupling of } \mu, \nu \}$$

The Wasserstein distance controls the Kolmogorov distance – at least when one of the measures has a bounded density.

Prop: If $d\nu = \rho\, dx$ and $\rho \leq C < \infty$, then for any $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

$$d_{Kol}(\mu, \nu) \leq 2\sqrt{C\, d_{W_1}(\mu, \nu)}.$$

Pf. Fix $t \in \mathbb{R}$, $\varepsilon > 0$, and define two continuous approximations to $\mathbb{1}_{(-\infty, t]}$:



$$\psi_- \leq \mathbb{1}_{(-\infty,t]} \leq \psi_+$$

$$\therefore \int \mathbb{1}_{(-\infty,t]}\, d\mu - \int \mathbb{1}_{(-\infty,t]}\, d\nu$$

$$\mu(-\infty,t] - \nu(-\infty,t]$$

$$\leq \int \varphi_t \, d\mu - \int \varphi_t \, d\nu + \int (\varphi_t - \mathbb{1}_{(-\infty,t]}) \, d\nu$$

$$\therefore \quad \mu(-\infty,t] - \nu(-\infty,t] \leq \frac{1}{\varepsilon} d_{W_1}(\mu,\nu) + C\varepsilon, \text{ for all } \varepsilon > 0.$$

optimize over $\varepsilon > 0$.

$$\therefore \leq 2\sqrt{C \, d_{W_1}(\mu,\nu)}$$

Now use φ_- to prove the reverse ineq. for $\nu(-\infty,t] - \mu(-\infty,t]$.

$$\therefore \sup_{t \in \mathbb{R}} |\mu(-\infty,t] - \nu(-\infty,t]| \leq 2\sqrt{C \, d_{W_1}(\mu,\nu)}$$

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As usual, we apply a metric on measures to random variables by $d(X,Y) := d(\mu_X, \mu_Y)$.

Cor: If $Z \overset{d}{=} \mathcal{N}(0,1)$, then for any random variable X,

$$d_{Kol}(X,Z) \leq 2\sqrt{d_{W_1}(X,Z)}$$

One of our goals is to prove (a version of) the:

Berry-Esseen Theorem:

Let $\{X_n\}_{n=1}^{\infty}$ be iid. L^3 random variables, with $\mathbb{E}[X_j]=0$, $\mathbb{E}[X_j^2]=1$, $\mathbb{E}[|X_j|^3]=\varrho^3$.

Let $S_n = X_1 + \cdots + X_n$. If $Z \overset{d}{=} \mathcal{N}(0,1)$, then

$$d_{Kol}\left(\frac{S_n}{\sqrt{n}}, Z\right) \leq C\frac{\varrho^3}{\sqrt{n}}$$