Here's an example application of the Lindberg-Feller / Liapunov CLT.

$\{X_k\}_{k=1}^{\infty}$ independent Bernoulli random variables,

$$X_k \overset{d}{=} \text{Bernoulli}\left(\tfrac{1}{k}\right)$$

$\mathbb{E}[X_k] =$

$\mathrm{Var}[X_k] = \mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2$

Set $\sigma_n^2 = \mathrm{Var}[X_1 + \cdots + X_n] = \sum_{k=1}^{n} \mathrm{Var}[X_k]$

Define $X_{n,k} = \frac{1}{\sigma_n} \mathring{X}_k = \frac{1}{\sigma_n}\left(X_k - \frac{1}{k}\right)$

$\sigma_{n,k}^2 = \mathrm{Var}[X_{n,k}] = \frac{1}{\sigma_n^2} \mathrm{Var}[X_k] \quad \therefore \quad \sum_{k=1}^{n} \sigma_{n,k}^2$

So $\{X_{n,k}\}_{k=1}^{n}$, $n \in \mathbb{N}$ is a standard triangular array.

Does it satisfy the Lindberg condition?

Prop: The Liapunov condition holds:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[X_{n,k}^4] = 0.$$

Pf.

$$\overset{\|}{\frac{1}{\sigma_n^4} \sum_{k=1}^{n} \mathbb{E}\left[\left(X_k - \frac{1}{k}\right)^4\right]}$$

$$\| $$

$$X_k^4 - 4X_k^3 \cdot \frac{1}{k} + 6X_k^2 \cdot \frac{1}{k^2} - 4X_k \cdot \frac{1}{k^3} + \frac{1}{k^4}$$

Since (Liap) $\Rightarrow$ (Lind), we conclude by the Lindberg-Feller CLT,

$$S_n = X_{n,1} + \cdots + X_{n,n} \xrightarrow{w} \mathcal{N}(0,1)$$

$$\parallel$$

$$\frac{\sum_{k=1}^{n} \left(X_k - \frac{1}{k}\right)}{\sigma_n} = \frac{\sum_{k=1}^{n} X_k - H(n)}{\sigma_n}$$

Thus $S_n \overset{d}{\sim} \mathcal{N}(\log n, \log n)$

This isn't just a contrived example,

Random Permutations

S_n = permutation group on n letters.

$\# S_n = n!$

Cycles: $\pi = \begin{Bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 4 & 3 & 1 \end{Bmatrix}$

Suppose we sample a permutation uniformly at random from S_n.

I.e. we use the measure $\mu\{\pi\} = \frac{1}{n!} \quad \forall \pi \in S_n$.

What can we say about the random variable

$C_n = \#\,\text{cycles}$

Feller Coupling

Given a sequence $(X_1, \ldots, X_n) \in \{0,1\}^n$, construct a random permutation π as follows.

- start @ 1 - If $X_n = 1$, (1) is a singleton cycle in π;
  - If $X_n = 0$, choose an element uniformly from $\{2, \ldots, n\}$ to include in the cycle with 1.

⋮

- At stage i - If $X_{n-i+1} = 1$, close the current cycle, and begin a new cycle with the smallest unused number.
  - If $X_{n-i+1} = 0$, choose an element uniformly from those yet unused to include in the current cycle.

Eg. $(X_1, \dots, X_6) = (1,1,0,1,0,0)$

1.

2 3 4 5 6

The permutation produced is random (unless $X_k = 1 \ \forall k$), but the number of cycles is determined: you close an old cycle and start a new one when $X_{n-i+1} = 1$, so $C_n =$

Theorem: (Feller, 1945)

If $(X_1, \dots, X_n)$ are independent Bernoulli random variables with $\mathbb{P}(X_k = 1) = \frac{1}{k}$, then the above procedure produces a uniformly random permutation.