Here's an example application of the Lindberg-Feller / Liapunov CLT.

$\{X_k\}_{k=1}^{\infty}$ independent Bernoulli random variables,

$$X_k \overset{d}{=} \text{Bernoulli}\left(\frac{1}{k}\right) \qquad \mathbb{P}(X_k=1)=\frac{1}{k}$$
$$\mathbb{P}(X_k=0)=1-\frac{1}{k}$$

$$\mathbb{E}[X_k] = \frac{1}{k}$$

$$\mathrm{Var}[X_k] = \mathbb{E}[\cancel{X_k^2}^{\,X_k}] - \mathbb{E}[X_k]^2 = \frac{1}{k} - \frac{1}{k^2}$$

$$H(n) \sim \log n + \gamma \leftarrow \text{Euler's constant} \doteq 0.5772\ldots$$

Set $\sigma_n^2 = \mathrm{Var}[X_1 + \cdots + X_n] = \sum_{k=1}^n \mathrm{Var}[X_k] = \underbrace{\sum_{k=1}^n \frac{1}{k}}_{=H(n)} - \sum_{k=1}^n \frac{1}{k^2} \to \pi^2/6$

Define $X_{n,k} = \frac{1}{\sigma_n}\mathring{X}_k = \frac{1}{\sigma_n}\left(X_k - \frac{1}{k}\right)$ $\in L^2$, independent, centered.

$$\sigma_{n,k}^2 = \mathrm{Var}[X_{n,k}] = \frac{1}{\sigma_n^2}\mathrm{Var}[X_k] \qquad \therefore \quad \sum_{k=1}^n \sigma_{n,k}^2 = 1.$$

So $\{X_{n,k}\}_{k=1}^n$, $n \in \mathbb{N}$ is a standard triangular array.

Does it satisfy the Lindberg condition?

Prop: The Liapunov condition holds:

$$\lim_{n\to\infty} \sum_{k=1}^{n} E[X_{n,k}^4] = 0.$$

Pf. $E[X_k] = \frac{1}{k}$

$$\| \qquad \frac{1}{\sigma_n^4} \sum_{k=1}^{n} E\left[\left(X_k - \frac{1}{k}\right)^4\right]$$

$$\|$$

$$E\left[X_k^{\cancel{4}} - 4X_k^{\cancel{3}} \cdot \frac{1}{k} + 6X_k^{\cancel{2}} \cdot \frac{1}{k^2} - 4X_k \cdot \frac{1}{k^3} + \frac{1}{k^4}\right]$$

$$= \frac{1}{k} - 4\frac{1}{k^2} + 6\frac{1}{k^3} - 4\frac{1}{k^4} + \frac{1}{k^4}$$

$$|E[\quad]| \leq \frac{1}{k} + \frac{15}{k^2}$$

$$\leq \frac{1}{\sigma_n^4}\left(\sum_{k=1}^{n} \frac{1}{k} + 15\sum_{k=1}^{n} \frac{1}{k^2}\right)$$

$< 30.$

$H(n) \sim \log n$

$$\lesssim \frac{1}{(\log n)^2}\left(\log n + 30\right) \to 0 \text{ as } n\to\infty.$$

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Since (Liap) $\Rightarrow$ (Lind), we conclude by the Lindberg-Feller CLT,

$$S_n = X_{n,1} + \cdots + X_{n,n} \xrightarrow{w} \mathcal{N}(0,1)$$

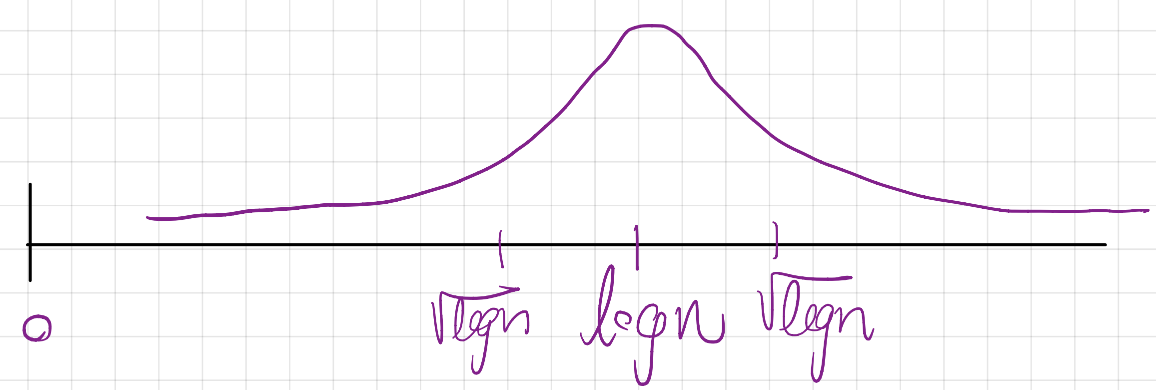
$$\|$$

$$\frac{\sum_{k=1}^{n} \left(X_k - \frac{1}{k}\right)}{\sigma_n} = \frac{\sum_{k=1}^{n} X_k - H(n)}{\sigma_n}$$

Thus $S_n \overset{d}{\sim} \mathcal{N}(\log n, \log n)$

σ_n^2

This isn't just a contrived example,



Random Permutations

S_n = permutation group on n letters.

$\# S_n = n!$

Cycles: $\pi = \begin{Bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 6 & 5 & 4 & 3 & 1 \end{Bmatrix}$

$1 \xrightarrow{\pi} 2 \xrightarrow{\pi} 6 \qquad 3 \rightleftarrows 5$

$\pi = (1\,2\,6)(3\,5)(4)$

Suppose we sample a permutation uniformly at random from S_n.

I.e. we use the measure $\mu\{\pi\} = \frac{1}{n!} \quad \forall \pi \in S_n$.

What can we say about the random variable

$C_n = \#$ cycles

$C_n(\pi) = 3$

$(1\,2 \cdots n)$

$1 \leq C_n \leq n \leftarrow (1)(2)\cdots(n)$

Feller Coupling

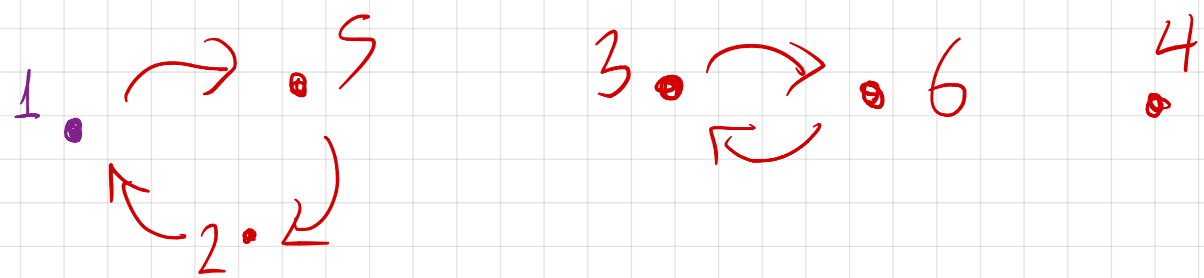
Given a sequence $(X_1, \ldots, X_n) \in \{0,1\}^n$, construct a random permutation π as follows.

- start @ 1 - If $X_n = 1$, (1) is a singleton cycle in π;
  - If $X_n = 0$, choose an element uniformly from $\{2, \ldots, n\}$ to include in the cycle with 1.

⋮

- At stage i - If $X_{n-i+1} = 1$, close the current cycle, and begin a new cycle with the smallest unused number.
  - If $X_{n-i+1} = 0$, choose an element uniformly from those yet unused to include in the current cycle.

Eg. $(X_1,\dots,X_6) = (1,1,0,1,0,0)$ $\rightsquigarrow$ $(1,5,2)(3,6)(4)$



~~2~~ 3 4 ~~5~~ ~~6~~

The permutation produced is random (unless $X_k=1\ \forall k$), but the number of cycles is determined: you close an old cycle and start a new one when $X_{n-i+1}=1$, so $C_n = X_1+X_2+\dots+X_n$.

Theorem: (Feller, 1945)

If $(X_1,\dots,X_n)$ are independent Bernoulli random variables with $\mathbb{P}(X_k=1)=\frac{1}{k}$, then the above procedure produces a uniformly random permutation.

$C_n \overset{d}{\sim} \mathcal{N}(\log n, \log n)$