Standard triangular array:

For each $n \in \mathbb{N}$, $\{X_{n,k}\}_{k=1}^{n}$ independent, L^2, centered

$\mathbb{E}[X_{n,k}] = 0$, $Var[X_{n,k}] = \sigma_{n,k}^2$, $\sum_{k=1}^{n} \sigma_{n,k}^2 = 1$.

$S_n = X_{n,1} + \cdots + X_{n,n}$ $\therefore Var[S_n] = 1$

Lindberg Condition (Lind)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[X_{n,k}^2 : |X_{n,k}| > \varepsilon] = 0 \quad \forall \varepsilon > 0.$$

Decaying Variances: (DV)

$$\max_{1 \le k \le n} \sigma_{n,k}^2 \to 0 \text{ as } n \to \infty$$

Lindberg-Feller CLT (Part 1)

If $\{X_{n,k}\}_{1\le k\le n}^{n\in\mathbb{N}}$ is a standard triangular array satisfying (Lind), then $S_n \to_w \mathcal{N}(0,1)$.

Lindberg-Feller CLT (Part 2)

If $\{X_{n,k}\}_{1\le k\le n}^{n\in\mathbb{N}}$ is a standard triangular array satisfying (DV), and if $S_n \to_w \mathcal{N}(0,1)$, then (Lind) holds.

Lemma 1: If $a_j, b_j \in \mathbb{C}$ with $|a_j|, |b_j| \le 1$, then

$$|a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n| \le \sum_{j=1}^{n} |a_j - b_j| .$$

Pf. Proceed by induction.

Let $a = a_1 \cdots a_{n-1}$, $b = b_1 \cdots b_{n-1}$ Then $|a|, |b| \le 1$.

$$|a a_n - b b_n|$$

Lemma 2: If $X \in L^2$, $|\varphi_X(\xi) - (1 + i\mathbb{E}[X]\xi - \frac{1}{2}\mathbb{E}[X^2]\xi^2)| \le \xi^2 \varepsilon(\xi)$

where $\varepsilon(\xi) = \mathbb{E}[\, X^2 \wedge \frac{|X|^3}{3!} |\xi| \,] \downarrow 0$ as $|\xi| \downarrow 0$.

Pf. Taylor's theorem: $|e^{it} - (1 + it - \frac{1}{2}t^2)| \le \frac{|it|^3}{3!}$.

Also:

Lindberg-Feller CLT (Part 1)

If $\{X_{n,k}\}_{1\leq k\leq n}^{n\in\mathbb{N}}$ is a standard triangular array satisfying (Lind), then $S_n \to_w \mathcal{N}(0,1)$.

Pf. Suffices to show $\varphi_{S_n}(\xi) \to e^{-\xi^2/2}$ $\forall \xi \in \mathbb{R}$.

By Lemma 1, $\left|\varphi_{S_n}(\xi) - e^{-\xi^2/2}\right| \leq \sum_{k=1}^{n} \left|\varphi_{X_{n,k}}(\xi) - \right.$

$$\left|\varphi_{X_{n,k}}(\xi) - e^{-\frac{\xi^2}{2}\sigma_{n,k}^2}\right| \leq \underbrace{\left|\varphi_{X_{n,k}}(\xi) - \left(1 - \tfrac{1}{2}\sigma_{n,k}^2\xi^2\right)\right|}_{A_{n,k}} + \underbrace{\left|\left(1 - \tfrac{1}{2}\sigma_{n,k}^2\xi^2\right) - e^{-\frac{\xi^2}{2}\sigma_{n,k}^2}\right|}_{B_{n,k}}$$

Suffices to show $\sum_{k=1}^{n}(A_{n,k} + B_{n,k}) \to 0$ as $n \to \infty$.

$$A_{n,k} = \left| \varphi_{X_{n,k}}(\xi) - \left(1 - \frac{1}{2}\sigma_{n,k}^2 \xi^2\right) \right| \leq \xi^2 \mathbb{E}\left[X_{n,k}^2 \wedge \frac{|\xi| \, |X_{n,k}|^3}{3!} \right]$$

$$\leq \xi^2 \Bigg(\mathbb{E}\left[X_{n,k}^2 \wedge \frac{|\xi|}{3!} |X_{n,k}|^3 : |X_{n,k}| \leq \varepsilon \right]$$

$$+ \mathbb{E}\left[X_{n,k}^2 \wedge \frac{|\xi|}{3!} |X_{n,k}|^3 : |X_{n,k}| > \varepsilon \right] \Bigg)$$

$$\therefore \sum_{k=1}^{n} A_{n,k} \leq \frac{|\xi|^3}{3!} \varepsilon \sum_{k=1}^{n} \sigma_{n,k}^2 + \xi^2 \sum_{k=1}^{n} \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > \varepsilon \right]$$

$$\therefore \limsup_{n \to \infty} \sum_{k=1}^{n} A_{n,k} \leq$$

$$B_{n,k} = \left| e^{-\frac{\xi^2}{2}\sigma_{n,k}^2} - \left(1 - \frac{1}{2}\sigma_{n,k}^2 \xi^2\right)\right|$$

Calculus estimate: $|e^{-u} - (1-u)| \le \frac{u^2}{2} \quad \forall u \ge 0$.

$$\therefore \sum_{k=1}^{n} B_{n,k} \le \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{2}\sigma_{n,k}^2 \xi^2\right)^2 = \frac{1}{8}\xi^4 \sum_{k=1}^{n} \sigma_{n,k}^4 .$$