Standard triangular array:

For each $n \in \mathbb{N}$, $\{X_{n,k}\}_{k=1}^{n}$ independent, L^2, centered

$\mathbb{E}[X_{n,k}] = 0$, $\mathrm{Var}[X_{n,k}] = \sigma_{n,k}^2$, $\sum_{k=1}^{n} \sigma_{n,k}^2 = 1$.

$S_n = X_{n,1} + \cdots + X_{n,n}$ $\therefore$ $\mathrm{Var}[S_n] = 1$

Lindberg Condition (Lind)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[X_{n,k}^2 : |X_{n,k}| > \varepsilon] = 0 \quad \forall \varepsilon > 0.$$

$\Rightarrow$

Decaying Variances: (DV)

$$\max_{1 \le k \le n} \sigma_{n,k}^2 \to 0 \text{ as } n \to \infty$$

Lindberg-~~Feller~~ CLT (Part 1)

If $\{X_{n,k}\}_{1\le k\le n}^{n\in\mathbb{N}}$ is a standard triangular array satisfying (Lind), then $S_n \to_w \mathcal{N}(0,1)$.

Lindberg-Feller CLT (Part 2)

If $\{X_{n,k}\}_{1\le k\le n}^{n\in\mathbb{N}}$ is a standard triangular array satisfying (DV), and if $S_n \to_w \mathcal{N}(0,1)$, then (Lind) holds.

[Driver §30.2]

Lemma 1: If $a_j, b_j \in \mathbb{C}$ with $|a_j|, |b_j| \leq 1$, then

$$|a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n| \leq \sum_{j=1}^{n} |a_j - b_j| .$$

Pf. Proceed by induction. (Base case: $n=1$, clear)

Let $a = a_1 \cdots a_{n-1}$, $b = b_1 \cdots b_{n-1}$. Then $|a|, |b| \leq 1$.

$$|a a_n - b b_n| = |a a_n - b a_n + b a_n - b b_n|$$

$$\leq |a-b| \underbrace{\cancel{|a_n|}}_{\leq 1} + \underbrace{\cancel{|b|}}_{\leq 1} |a_n - b_n|$$

by ind. hyp.

$$\leq \sum_{j=1}^{n-1} |a_j - b_j| + |a_n - b_n|$$

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Lemma 2: If $X \in L^2$, $|\varphi_X(\xi) - (1 + i\mathbb{E}[X]\xi - \frac{1}{2}\mathbb{E}[X^2]\xi^2)| \leq \xi^2 \varepsilon(\xi)$

where $\varepsilon(\xi) = \mathbb{E}[X^2 \wedge \frac{|X|^3}{3!}|\xi|] \downarrow 0$ as $|\xi| \downarrow 0$. by DCT.

Pf. Taylor's theorem: $|e^{it} - (1 + it - \frac{1}{2}t^2)| \leq \frac{|it|^3}{3!}$

Also: $|e^{it} - (1+it)| + \frac{1}{2}t^2 \leq \frac{1}{2}t^2 + \frac{1}{2}t^2 = t^2$.

$\therefore \mathbb{E}[|e^{i\xi X} - (1 + i\xi X - \frac{1}{2}\xi^2 X^2)|] \leq \mathbb{E}[\frac{|\xi X|^3}{3!} \wedge (\xi X)^2]$ ///

Lindberg-Feller CLT (Part 1)

If $\{X_{n,k}\}_{1\le k\le n}^{n\in\mathbb{N}}$ is a standard triangular array satisfying (Lind), then $S_n \to_w \mathcal{N}(0,1)$.

Pf. Suffices to show $\varphi_{S_n}(\xi) \to e^{-\xi^2/2}$ $\forall \xi\in\mathbb{R}$.

$$\varphi_{S_n}(\xi) = \varphi_{X_{n,1}}(\xi)\cdots\varphi_{X_{n,n}}(\xi)$$

$$e^{-\xi^2/2} = e^{-\frac{\xi^2}{2}\sigma_{n,1}^2}\cdots e^{-\frac{\xi^2}{2}\sigma_{n,n}^2} \quad b/c \quad \sum_{k=1}^n \sigma_{n,k}^2 = 1$$

By Lemma 1, $\left|\varphi_{S_n}(\xi) - e^{-\xi^2/2}\right| \le \sum_{k=1}^n \left|\varphi_{X_{n,k}}(\xi) - e^{-\frac{\xi^2}{2}\sigma_{n,k}^2}\right|$

$$\varphi_{X_{n,k}}(\xi) \approx 1 + i\,\cancel{\mathbb{E}[X_{n,k}]}_{0}\,\xi - \frac{1}{2}\underline{\mathbb{E}[X_{n,k}^2]}\,\xi^2 = 1 - \frac{1}{2}\sigma_{n,k}^2\xi^2$$

$$\left|\varphi_{X_{n,k}}(\xi) - e^{-\frac{\xi^2}{2}\sigma_{n,k}^2}\right| \le \underbrace{\left|\varphi_{X_{n,k}}(\xi) - \left(1-\tfrac{1}{2}\sigma_{n,k}^2\xi^2\right)\right|}_{A_{n,k}} + \underbrace{\left|\left(1-\tfrac{1}{2}\sigma_{n,k}^2\xi^2\right) - e^{-\frac{\xi^2}{2}\sigma_{n,k}^2}\right|}_{B_{n,k}}$$

Suffices to show $\sum_{k=1}^n (A_{n,k}+B_{n,k}) \to 0$ as $n\to\infty$.

$$A_{n,k} = \left| \varphi_{X_{n,k}}(\xi) - \left(1 - \tfrac{1}{2}\sigma_{n,k}^2 \xi^2\right) \right| \underset{\text{Lemma 2}}{\leq} \xi^2 \, \mathbb{E}\left[X_{n,k}^2 \wedge \frac{|\xi|\,|X_{n,k}|^3}{3!}\right]$$

$$\leq \xi^2 \Big(\mathbb{E}\left[\cancel{X_{n,k}^2} \wedge \frac{|\xi|}{3!}|X_{n,k}|^3 : |X_{n,k}| \leq \varepsilon\right]$$

$$+ \mathbb{E}\left[X_{n,k}^2 \wedge \cancel{\frac{|\xi|}{3!}|X_{n,k}|^3} : |X_{n,k}| > \varepsilon\right] \Big) \qquad \text{for } \varepsilon > 0.$$

$$\leq \frac{|\xi|^3}{3} \varepsilon \, \underbrace{\mathbb{E}\left[|X_{n,k}|^2 \cancel{: |X_{n,k}| \leq \varepsilon}\right]}_{\sigma_{n,k}^2} + \xi^2 \, \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > \varepsilon\right]$$

$$\therefore \sum_{k=1}^{n} A_{n,k} \leq \frac{|\xi|^3}{3!} \varepsilon \underbrace{\sum_{k=1}^{n} \sigma_{n,k}^2}_{= 1} + \xi^2 \underbrace{\sum_{k=1}^{n} \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > \varepsilon\right]}_{\to 0 \text{ as } n \to \infty}$$

$$\therefore \limsup_{n \to \infty} \sum_{k=1}^{n} A_{n,k} \leq \frac{|\xi|^3}{6} \varepsilon. \qquad \forall \varepsilon > 0.$$

$$\therefore \quad = 0.$$

$$B_{n,k} = \left| e^{-\frac{\xi^2}{2}\sigma_{n,k}^2} - \left(1 - \tfrac{1}{2}\sigma_{n,k}^2 \xi^2\right)\right|$$

Calculus estimate: $\left| e^{-u} - (1-u) \right| \le \frac{u^2}{2} \quad \forall u \ge 0.$

$$\therefore \sum_{k=1}^n B_{n,k} \le \sum_{k=1}^n \frac{1}{2}\left(\frac{1}{2}\sigma_{n,k}^2 \xi^2\right)^2 = \frac{1}{8}\xi^4 \sum_{k=1}^n \underbrace{\sigma_{n,k}^4}_{\sigma_{n,k}^2 \cdot \sigma_{n,k}^2 \le \max_{1\le j\le n} \sigma_{n,j}^2 \cdot \sigma_{n,k}^2}.$$

$$\le \frac{1}{8}\xi^4 \underbrace{\max_{1\le j\le n}\sigma_{n,j}^2}_{\xrightarrow{n\to\infty} 0 \text{ by (DV)} \Leftarrow \text{(Lind)}} \cdot \underbrace{\sum_{k=1}^n \sigma_{n,k}^2}_{1}$$

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