The CLT arises from independence.
Identical distribution is <u>not</u> strictly required,
but some kind of "average uniformity" <u>is</u> needed.

E.g. $X_n \overset{d}{=} \mathcal{N}(0, b_n^2)$ independent. Let $S_n = X_1 + \cdots + X_n$.

If all $b_n = b$, or even if $b_n \to b$,
But what if, e.g., $b_n = \sqrt{n}$? Then $\sum_{j=1}^{n} b_n^2 = \frac{1}{2} n(n+1)$.

# Triangular Arrays

$\{X_{n,k}\}_{k=1}^{n}$ independent, centered $L^2$ random variables
$\mathbb{E}[X_{n,k}]=0$, $\mathbb{E}[X_{n,k}^2] = Var[X_{n,k}] =: \sigma_{n,k}^2 < \infty$

$$\begin{array}{lllll}
X_{1,1} & & & & \\
X_{2,1} & X_{2,2} & & & \\
X_{3,1} & X_{3,2} & X_{3,3} & & \\
\vdots & \vdots & \vdots & \ddots & \\
X_{n,1} & X_{n,2} & X_{n,3} & \cdots & X_{n,n} \\
\vdots & \vdots & \vdots & & \ddots
\end{array}$$

Better to build in standardization.

Assume $\sum_{k=1}^{n} \sigma_{n,k}^2 = 1$.

Eg. If $\{X_n\}_{n=1}^{\infty}$ are iid with $Var[X_n] = b^2$, the CLT deals with

$$\frac{1}{b\sqrt{n}} \sum_{k=1}^{n} \overset{\circ}{X}_k .$$

Why?

# Average Uniformity Conditions

$\{X_{n,k}\}_{k=1}^{n}$ centered $L^2$ random variables

$\mathbb{E}[X_{n,k}]=0$, $\mathbb{E}[X_{n,k}^2] = Var[X_{n,k}] =: \sigma_{n,k}^2$, $\sum_{k=1}^{n} \sigma_{n,k}^2 = 1$.

Below are two conditions that precisely interpret the requirement that "the terms are small and comparable in size"

- (DV) The Decaying Variance condition:

$$\max_{1\le k\le n} \sigma_{n,k}^2 \to 0 \quad \text{as } n\to\infty$$

- (UAN) The Uniform Asymptotic Negligibility condition:

$$\forall \varepsilon>0, \ \lim_{n\to\infty} \max_{1\le k\le n} \mathbb{P}(|X_{n,k}|>\varepsilon) = 0.$$

**Lemma**: (DV) $\Rightarrow$ (UAN).

Pf. By Chebyshev, $\mathbb{P}(|X_{n,k}|>\varepsilon) \le \dfrac{Var[X_{n,k}]}{\varepsilon^2}$

We'd like to prove a CLT for triangular arrays, assuming something like (DV). That's not quite correct. We need slightly stronger conditions.

- (Lind) The Lindberg condition:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[X_{n,k}^2 : |X_{n,k}| > \varepsilon] = 0 \quad \forall \varepsilon > 0.$$

This is a decreasing function of $\varepsilon > 0$, so suffices for this to hold for small $\varepsilon > 0$.

Eg. $X_{n,k} = \frac{1}{b\sqrt{n}} \mathring{X}_k$ where $\{X_k\}_{k=1}^{\infty}$ are iid w $\mathrm{Var}[X_k] = b^2$.

$$\sum_{k=1}^{n} \mathbb{E}[X_{n,k}^2 : |X_{n,k}| > \varepsilon] \quad \frac{1}{b^2 n} \sum_{k=1}^{n} \mathbb{E}[X_k^2 : |X_k| > |b|\sqrt{n}\,\varepsilon]$$

**Prop:** (Lind) $\Rightarrow$ (DV).

**Pf.** $\sigma_{n,k}^2 = \mathbb{E}[X_{n,k}^2] = \mathbb{E}[X_{n,k}^2(\mathbb{1}_{|X_{n,k}|\le\varepsilon} + \mathbb{1}_{|X_{n,k}|>\varepsilon})]$

$$= \mathbb{E}[X_{n,k}^2 : |X_{n,k}|\le\varepsilon] + \mathbb{E}[X_{n,k}^2 : |X_{n,k}|>\varepsilon]$$

## Lindberg-Feller CLT

$\{X_{n,k}\}_{k=1}^n$ centered independent $L^2$ random variables, $S_n = X_{n,1} + \cdots + X_{n,n}$

$\mathbb{E}[X_{n,k}]=0$, $\mathbb{E}[X_{n,k}^2] = Var[X_{n,k}] =: \sigma_{n,k}^2$, $\sum_{k=1}^n \sigma_{n,k}^2 = 1$. (ie. $Var[S_n]=1$)

If the Lindberg condition holds for $\{X_{n,k}\}$, then $S_n \to_w \mathcal{N}(0,1)$.

Conversely: if $\{X_{n,k}\}$ satisfy the decaying variance condition (DV) and if $S_n \to_w \mathcal{N}(0,1)$, then the Lindberg condition holds for $\{X_{n,k}\}$.

Historical note: before Lindberg-Feller, a "triangular CLT" was proved by Liapunov, with an even stronger hypothesis.

- (Liap) $\exists \alpha > 2$ s.t. $\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[\,|X_{n,k}|^{\alpha}] = 0$

or a bit weaker:

$\exists\ \phi : [0,\infty) \to [0,\infty)$ non-decreasing, $\phi(t) > 0\ \forall t > 0$, s.t.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}[\,X_{n,k}^{2}\,\phi(X_{n,k})] = 0$$

Lemma: (Liap) $\Rightarrow$ (Lind)

Pf. $\sum_{k=1}^{n} \mathbb{E}[\,X_{n,k}^{2} : |X_{n,k}| > \varepsilon]$