What's so special about Gaussians? One answer:

Def: A probability measure $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is infinitely divisible if, for each $n \in \mathbb{N}$, $\exists\, \mu_n \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $\mu = \mu_n^{*n}$

I.e. $\exists\, \{X_{n,k}\}_{k=1}^{n}$ iid s.t. $S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mu$

I.e. $\exists$ non-constant characteristic function φ_n s.t. $\hat{\mu}(\xi) = \varphi_n(\xi)^n \ \forall \xi \in \mathbb{R}$.

Eg. If $X_{n,k} \overset{d}{=} \mathcal{N}(0, \sigma^2/n)$ are independent, then

$$X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mathcal{N}(0, \sigma^2)$$

Eg. If $N_{n,k} \overset{d}{=} \mathrm{Poisson}(\lambda/n)$ are independent, $S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mathrm{Poisson}(\lambda)$.

Note: if μ, ν are infinitely divisible,

so is $\mu * \nu$

Eg. If $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$, then $\hat{\mu}(\xi) = \cos\xi$.

Suppose $X_1 \overset{d}{=} X_2$, independent, s.t. $X_1 + X_2 \overset{d}{=} \mu$.

$$\cos\xi = \hat{\mu}(\xi) = \hat{\mu}_{X_1}(\xi)^2$$

Theorem: A probability measure $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is infinitely divisible iff $\exists$ a "triangular array"

$$\{X_{n,k}\}_{k=1}^{m_n} \qquad m_n \uparrow \infty, \quad n \in \mathbb{N}$$

of random variables s.t. for each n, $\{X_{n,k}\}_{k=1}^{m_n}$ are i.i.d., and

$$S_n = \sum_{k=1}^{m_n} X_{n,k} \xrightarrow{w} X \overset{d}{=} \mu .$$

Pf. $(\Rightarrow)$ If μ is infinitely divisible, can find $\{X_{n,k}\}_{k=1}^{n}$ iid s.t. $S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mu$

$(\Leftarrow)$ Step 1 is to show that if such a triangular array exists, so does one with $m_n = n$.

We'll skip this step, which involves some kind of involved tail bound estimates. We'll prove the theorem with $m_n = n$.

We have $\{X_{n,k}\}_{k=1}^{n}$ iid for each n s.t.

$$S_n = X_{n,1} + \cdots + X_{n,n} \to_w X \stackrel{d}{=} \mu .$$

Fix $\ell \in \mathbb{N}$. Consider $S_{n\ell} = \sum_{k=1}^{n\ell} X_{n,k} = \sum_{i=1}^{\ell} S_n^i$

Since $S_{n\ell} \xrightarrow[w]{n\to\infty} X$, we know that $\{\mu_{S_{n\ell}}\}_{n\in\mathbb{N}}$ is tight.

$$\sup_{n\in\mathbb{N}} \mathbb{P}(|S_{n\ell}| > r)$$

$$\mathbb{P}(S_n^1 > r)^\ell = \mathbb{P}(S_n^1 > r, \ldots, S_n^\ell > r)$$

Similarly $\mathbb{P}(-S_n^1 > r)^\ell$ $\qquad \therefore \mathbb{P}(|S_n^1| > r)$

So $\{S_n^1\}_{n=1}^\infty$ is tight, and $\therefore$ by Helly/Prokhorov,
$\exists$ subsequence $\{n_j\}_{j=1}^\infty$ with $S_{n_j}^1 \xrightarrow[w]{j\to\infty} Y$

$\therefore$ since $S_n^1, \dots, S_n^\ell$ are iid, can select $Y_1, \dots, Y_\ell$ iid.

s.t. $S_{n_j}^i \xrightarrow[w]{j\to\infty} Y_i$

$$\therefore \quad S_{n_j\ell} = \sum_{i=1}^{\ell} S_{n_j}^i \xrightarrow[w]{j\to\infty} Y_1 + \cdots + Y_\ell$$

It turns out we can even weaken the iid condition on the "rows" of the triangular array. We'll explore this in the Gaussian case.