What's so special about Gaussians? One answer:

Def: A probability measure $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is **infinitely divisible** if, for each $n \in \mathbb{N}$,
$\exists\, \mu_n \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t. $\mu = \mu_n^{*n} = \mu_n * \mu_n * \cdots * \mu_n$.

I.e. $\exists\, \{X_{n,k}\}_{k=1}^{n}$ iid s.t. $S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mu$

I.e. $\exists$ non-constant characteristic function φ_n s.t. $\hat{\mu}(\xi) = \varphi_n(\xi)^n \;\; \forall \xi \in \mathbb{R}$.

Eg. If $X_{n,k} \overset{d}{=} \mathcal{N}(0, \sigma^2/n)$ are independent, then

$$S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mathcal{N}(0, \sigma^2)$$

$X_{n,k} \overset{d}{=} \frac{\sigma}{\sqrt{n}} Z$ for $Z \overset{d}{=} \mathcal{N}(0,1)$ $\quad\therefore\; \varphi_{X_{n,k}}(\xi) = \varphi_Z\left(\frac{\sigma}{\sqrt{n}}\xi\right) = e^{-\frac{\sigma^2}{n}\xi^2/2}$

$$\therefore\; \varphi_{S_n}(\xi) = \left(e^{-\frac{\sigma^2}{\cancel{n}}\frac{\xi^2}{2}}\right)^{\cancel{n}} = e^{-\sigma^2\xi^2/2} = \varphi_Z(\sigma\xi) = \varphi_{\sigma Z}(\xi)$$

$$\Rightarrow S_n \overset{d}{=} \sigma Z \overset{d}{=} \mathcal{N}(0, \sigma^2).$$

Eg. If $N_{n,k} \overset{d}{=} \mathrm{Poisson}(\lambda/n)$ are independent, $S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mathrm{Poisson}(\lambda)$.

Note: if μ, ν are infinitely divisible,

so is $\mu * \nu = (\mu_n * \cdots * \mu_n) * (\nu_n * \cdots * \nu_n)$

$= (\mu_n * \nu_n)^{*n}$

Eg. If $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$, then $\hat{\mu}(\xi) = \cos\xi$.

Suppose $X_1 \overset{d}{=} X_2$, independent, s.t. $X_1 + X_2 \overset{d}{=} \mu$. $\leftarrow$ supp μ $[-1,1]$

$\mathbb{P}(X_1 < -\frac{1}{2})^2 \le \mathbb{P}(X_1 + X_2 < -1) = 0 = \mathbb{P}(X_1 + X_2 > 1) \geq \mathbb{P}(X_1 > \frac{1}{2})^2$

$\{X_1 < -\frac{1}{2}\} \cap \{X_2 < -\frac{1}{2}\} \subseteq \{X_1 + X_2 < -1\}$ $\qquad$ $\{X_1 + X_2 > 1\} \supseteq \{X_1 > \frac{1}{2}\} \cap \{X_2 > \frac{1}{2}\}$

$\Rightarrow |X_1|, |X_2| \le \frac{1}{2}$ a.s.

$\therefore \mu_{X_1} = \mu_{X_2}$ has finite moments of all orders.

$\therefore \hat{\mu}_{X_1} = \hat{\mu}_{X_2} \in C^\infty$.

$\cos\xi = \hat{\mu}(\xi) = \hat{\mu}_{X_1}(\xi)^2$

$= f(\xi)^2$ $\quad f \in C^\infty(\mathbb{R})$

$0 = \cos\frac{\pi}{2} = f(\frac{\pi}{2})^2$

$\cos\xi = f(\xi)^2$

$\Rightarrow -\sin\xi = 2 f(\xi) f'(\xi)$

$-1 = -\sin\frac{\pi}{2} = 2 \cdot 0 \cdot f'(\xi)$. $\lightning$

Theorem: A probability measure $\mu \in \text{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is infinitely divisible iff $\exists$ a "triangular array"

$$\{X_{n,k}\}_{k=1}^{m_n} \qquad m_n \uparrow \infty, \quad n \in \mathbb{N}$$

of random variables s.t. for each n, $\{X_{n,k}\}_{k=1}^{m_n}$ are i.i.d., and

$$S_n = \sum_{k=1}^{m_n} X_{n,k} \xrightarrow{w} X \overset{d}{=} \mu .$$

Don't need equality, only weak convergence.

Pf. ($\Rightarrow$) If μ is infinitely divisible, can find $\{X_{n,k}\}_{k=1}^{n}$ iid s.t. $S_n = X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mu$

($\Leftarrow$) Step 1 is to show that if such a triangular array exists, so does one with $m_n = n$.

We'll skip this step, which involves some kind of involved tail bound estimates. We'll prove the theorem with $m_n = n$.

We have $\{X_{n,k}\}_{k=1}^{n}$ iid for each n s.t.

$$S_n = X_{n,1} + \cdots + X_{n,n} \to_w X \overset{d}{=} \mu.$$

Fix $l \in \mathbb{N}$. Consider $S_{nl} = \sum_{k=1}^{nl} X_{nl,k} = \sum_{i=1}^{l} S_n^i$

iid for $1 \le i \le l$ $\quad S_n^i = \sum_{j=n(i-1)+1}^{ni} X_{nl,j}$

Since $S_{nl} \xrightarrow[w]{n\to\infty} X$, we know that $\{\mu_{S_{nl}}\}_{n\in\mathbb{N}}$ is tight.

$\sup_{n\in\mathbb{N}} \mathbb{P}(|S_{nl}| > r)$ $\quad$ can make $<$ any ε with suff. large $r > 0$.

$\hookrightarrow$ decreasing as $r \uparrow \infty$.

$\le \varepsilon(r) \downarrow 0$ as $r \uparrow \infty$.

$$\mathbb{P}(S_n^1 > r)^l = \mathbb{P}(S_n^1 > r, \ldots, S_n^l > r) \le \mathbb{P}(S_{nl} > lr)$$

$\Rightarrow S_{nl} > lr.$ $\quad \le \mathbb{P}(|S_{nl}| > lr) \le \varepsilon(lr)$

Similarly $\mathbb{P}(-S_n^1 > r)^l \le \varepsilon(lr)$ $\quad \therefore \mathbb{P}(|S_n^1| > r) \le \varepsilon(lr)^{1/l} \downarrow 0$ as $r \uparrow \infty$

$\therefore \{\mu_{S_n^i}\}_{n=1}^{\infty}$ is tight.

So $\{S_n^1\}_{n=1}^\infty$ is tight, and $\therefore$ by Helly/Prokhorov,
$\exists$ subsequence $\{n_j\}_{j=1}^\infty$ with $S_{n_j}^1 \xrightarrow[w]{j\to\infty} Y$ for some Y.
$\therefore$ since $S_n^1, \dots, S_n^\ell$ are iid, can select $Y_1, \dots, Y_\ell$ iid.
s.t. $S_{n_j}^i \xrightarrow[w]{j\to\infty} Y_i$

$$\therefore \quad \underbrace{S_{n_j\ell}}_{\xrightarrow{w} X} = \sum_{i=1}^{\ell} S_{n_j}^i \xrightarrow[w]{j\to\infty} Y_1 + \dots + Y_\ell$$

$$\varphi_{S_{n_j}^1 + \dots + S_{n_j}^\ell}(\xi) = \varphi_{S_{n_j}^1}(\xi) \cdots \varphi_{S_{n_j}^\ell}(\xi) \qquad \forall \xi \in \mathbb{R}.$$

$$\xrightarrow{j\to\infty} \varphi_{Y_1}(\xi) \cdots \varphi_{Y_\ell}(\xi) = \varphi_{Y_1+\dots+Y_\ell}(\xi)$$

$$X \overset{d}{=} \underbrace{Y_1 + \dots + Y_\ell}_{\text{iid.}} \Rightarrow \mu_X \text{ is inf. div.} \quad ///$$

It turns out we can even weaken the iid. condition on the "rows" of the triangular array. We'll explore this in the Gaussian case.