Theorem: [26.15] (L^2-SLLN)

Let $\{X_n\}_{n=1}^{\infty}$ be independent L^2 random variables, with common mean $E[X_n] = t$. Let $S_n = X_1 + \cdots + X_n$, and let $b_n > 0$ s.t. $\sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$.

Then $\frac{\mathring{S}_n}{b_n} = \frac{S_n - nt}{b_n} \to 0$ a.s. and in L^2.

So, for example, if $\alpha > \frac{1}{2}$,

Theorem: (Basic Central Limit Theorem)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. L^2 random variables, with common mean $E[X_n] = t$, and variance $Var[X_n] = \sigma^2$.

Let $S_n = X_1 + \cdots + X_n$. Then $\frac{\mathring{S}_n}{\sigma\sqrt{n}} = \frac{S_n - nt}{\sigma\sqrt{n}} \to_w Z \overset{d}{=} \mathcal{N}(0,1)$.

Pf. By Lévy's continuity theorem, it suffices to show that

$$\varphi_{\mathring{S}_n/\sigma\sqrt{n}}(\xi) \to e^{-\xi^2/2} \quad \forall \xi \in \mathbb{R}$$

$X_1 \in L^2$, so $\mathbb{E}[\mathring{X}_1^2] = Var[X_1] = \sigma^2 < \infty$, $\therefore \varphi_{\mathring{X}_1} \in C^2$.

By Taylor's theorem,

$$\varphi_{\mathring{X}_1}(x) = \varphi_{\mathring{X}_1}(0) + \varphi_{\mathring{X}_1}'(0)\,x + \frac{1}{2}\varphi_{\mathring{X}_1}''(r(x))\,x^2$$

$$\therefore \varphi_{\mathring{X}_1}(\xi/\sigma\sqrt{n}) = 1 + \frac{1}{2}\varphi_{\mathring{X}_1}''(r(\xi/\sigma\sqrt{n}))\left(\frac{\xi}{\sigma\sqrt{n}}\right)^2$$

There is a similar CLT for iid random vectors, with any given (common) covariance of entries.

Def: Let Q be a positive semidefinite $d\times d$ matrix

i.e. $Q = AA^T$ for some $d\times d$ matrix A.

The centered normal distribution of covariance Q is the unique measure $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\hat{\mu}(\xi) = e^{-\frac{1}{2}Q\xi\cdot\xi}$

Denote it as $\mathcal{N}(0,Q)$.

Exercise: If $X \overset{d}{=} \mathcal{N}(0,Q)$, then $\mathrm{Cov}[X_i, X_j] = Q_{ij}$, and $X_i \overset{d}{=} \mathcal{N}(0, Q_{ii})$.

Theorem: (Multivariate CLT) If $\{X_n\}_{n=1}^{\infty}$ are iid random vectors in $\mathbb{R}^d$ with L^2 entries, and $Q = \mathbb{E}[\mathring{X}_1 \mathring{X}_1^T]$, then $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathring{X}_j \to_w Z \overset{d}{=} \mathcal{N}(0,Q)$.

Lemma: (Cramér-Wold Device)

Let $\{X_n\}_{n=1}^{\infty}$ and X be random vectors in $\mathbb{R}^d$.

Then $X_n \to_w X$ iff $\xi \cdot X_n \to_w \xi \cdot X \quad \forall \xi \in \mathbb{R}^d$.

Pf. If $\xi \cdot X_n \to_w \xi \cdot X$ the $\exp(i\xi \cdot X_n) \to_w \exp(i\xi \cdot X)$

$\therefore \mathbb{E}[f(e^{i\xi \cdot X_n})] \to \mathbb{E}[f(e^{i\xi \cdot X})] \quad \forall f \in C_b(\mathbb{C})$

Conversely, if $X_n \to_w X$, $\xi \in \mathbb{R}^d$, then for any $u \in \mathbb{R}$,

$\varphi_{\xi \cdot X_n}(u)$

Theorem: (Multivariate CLT) If $\{\mathbf{X}_n\}_{n=1}^{\infty}$ are iid random vectors in $\mathbb{R}^d$ with L^2 entries, and $Q = \mathbb{E}[\mathring{\mathbf{X}}_1 \mathring{\mathbf{X}}_1^T]$, then $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathring{\mathbf{X}}_j \to_w Z \stackrel{d}{=} \mathcal{N}(0, Q)$.

Pf. Fix $\xi \in \mathbb{R}^d$. Let $X_n^{\xi} := \xi \cdot \mathbf{X}_n$. Then $\{X_n^{\xi}\}_{n=1}^{\infty}$ are independent, and

$$\varphi_{X_n^{\xi}}(u) = \mathbb{E}[e^{iu\xi\cdot\mathbf{X}_n}]$$

$\therefore \{X_n^{\xi}\}_{n=1}^{\infty}$ are iid. They are in L^2:

$\mathbb{E}[X_n^{\xi}]$

$\mathrm{Var}[X_n^{\xi}]$

$\therefore$ By basic CLT, $\frac{1}{\sqrt{Q\xi\cdot\xi}} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (X_j^{\xi} - \xi\cdot\mathbb{E}[\mathbf{X}_1]) \to_w \mathcal{N}(0,1)$