**Theorem:** [26.15] ($L^2$-SLLN)

Let $\{X_n\}_{n=1}^{\infty}$ be independent $L^2$ random variables, with common mean $E[X_n] = t$. Let $S_n = X_1 + \cdots + X_n$, and let $b_n > 0$ s.t. $\sum_{n=1}^{\infty} \frac{1}{b_n^2} < \infty$.

Then $\frac{\mathring{S}_n}{b_n} = \frac{S_n - nt}{b_n} \to 0$ a.s. and in $L^2$.

So, for example, if $\alpha > \frac{1}{2}$, $\sum_{n=1}^{\infty} \frac{1}{(n^{\alpha})^2} = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} < \infty$ $\leftarrow$ But $\sum_{n=1}^{\infty} \frac{1}{(\sqrt{n})^2} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\therefore \frac{\mathring{S}_n}{n^{\alpha}} \to 0$ a.s. and in $L^2$ $\quad \frac{\mathring{S}_n}{n^{\alpha}} \to_w 0$.

**Theorem:** (Basic Central Limit Theorem)

Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. $L^2$ random variables, with common mean $E[X_n] = t$, and variance $Var[X_n] = \sigma^2$.

Let $S_n = X_1 + \cdots + X_n$. Then $\frac{\mathring{S}_n}{\sigma\sqrt{n}} = \frac{S_n - nt}{\sigma\sqrt{n}} \to_w Z \overset{d}{=} N(0,1)$.

Pf. By Lévy's continuity theorem, it suffices to show that

$$\varphi_{\mathring{S}_n/\sigma\sqrt{n}}(\xi) \to e^{-\xi^2/2} \quad \forall \xi \in \mathbb{R}$$

$$\| $$

$$\varphi_{\mathring{S}_n}(\xi/\sigma\sqrt{n}) = \varphi_{\mathring{X}_1+\cdots+\mathring{X}_n}(\xi/\sigma\sqrt{n}) = \varphi_{\mathring{X}_1}(\xi/\sigma\sqrt{n})^n$$

$X_1 \in L^2$, so $\mathbb{E}[\mathring{X}_1^2] = Var[X_1] = \sigma^2 < \infty$, $\therefore \varphi_{\mathring{X}_1} \in C^2$.

By Taylor's theorem,

$$\varphi_{\mathring{X}_1}(x) = \underbrace{\varphi_{\mathring{X}_1}(0)}_{=1} + \underbrace{\varphi'_{\mathring{X}_1}(0)}_{=\mathbb{E}[i\mathring{X}_1 \cancel{e^{i\xi\mathring{X}_1}}]|_{\xi=0} = 0} x + \frac{1}{2}\varphi''_{\mathring{X}_1}(r(x))\,x^2$$

for some $r(x)$ between $0$ and $x$. $\lim_{x\to 0} r(x) = 0$.

$$= 1 + \frac{1}{2}\varphi''_{\mathring{X}_1}(r(x))\,x^2$$

$$\therefore \left(\varphi_{\mathring{X}_1}(\xi/\sigma\sqrt{n})\right)^n = \left(1 + \frac{1}{2}\varphi''_{\mathring{X}_1}(r(\xi/\sigma\sqrt{n}))\left(\frac{\xi}{\sigma\sqrt{n}}\right)^2\right)^n$$

$$\to \varphi''_{\mathring{X}_1}(0) = \mathbb{E}[(i\mathring{X}_1)^2 \cancel{e^{i\xi\mathring{X}_1}}]|_{\xi=0} = -Var(X_1) = -\sigma^2$$

$$\lim_{n\to\infty} \varphi_{\mathring{X}_1}(\xi/\sigma\sqrt{n})^n = \lim_{n\to\infty}\left(1 + \frac{1}{2}(-\cancel{\sigma^2})\left(\frac{\xi^2}{\cancel{\sigma^2} n}\right)\right)^n = e^{-\xi^2/2}$$

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There is a similar CLT for iid random vectors, with any given (common) covariance of entries.

**Def:** Let $Q$ be a positive semidefinite $d \times d$ matrix (symmetric, all eigenvalues $\geq 0$)

i.e. $Q = AA^T$ for some $d \times d$ matrix $A$.

The centered normal distribution of covariance $Q$ is the unique measure $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\hat{\mu}(\xi) = e^{-\frac{1}{2} Q\xi \cdot \xi} = e^{-\frac{1}{2}|A\xi|^2}$

Denote it as $\mathcal{N}(0, Q)$.

**Exercise:** If $\mathbf{X} \overset{d}{=} \mathcal{N}(0,Q)$, then $\underbrace{\mathrm{Cov}[X_i, X_j] = Q_{ij}}_{Q = \mathbb{E}[\mathbf{X}\mathbf{X}^T]}$, and $X_i \overset{d}{=} \mathcal{N}(0, Q_{ii})$.

**Theorem:** (Multivariate CLT) If $\{\mathbf{X}_n\}_{n=1}^{\infty}$ are iid random vectors in $\mathbb{R}^d$ with $L^2$ entries, and $Q = \mathbb{E}[\mathring{\mathbf{X}}_1 \mathring{\mathbf{X}}_1^T]$, then $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathring{\mathbf{X}}_j \to_w Z \overset{d}{=} \mathcal{N}(0, Q)$.

**Lemma:** (Cramér-Wold Device)

Let $\{X_n\}_{n=1}^{\infty}$ and $X$ be random vectors in $\mathbb{R}^d$.

Then $X_n \to_w X$ iff $\xi \cdot X_n \to_w \xi \cdot X \quad \forall \xi \in \mathbb{R}^d$.

Pf. If $\xi \cdot X_n \to_w \xi \cdot X$ the $\exp(i\xi \cdot X_n) \to_w \exp(i\xi \cdot X)$

$$\therefore \mathbb{E}[f(e^{i\xi \cdot X_n})] \to \mathbb{E}[f(e^{i\xi \cdot X})] \quad \forall f \in C_b(\mathbb{C})$$

$$\mathbb{E}[e^{i\xi \cdot X_n}] \qquad \mathbb{E}[e^{i\xi \cdot X}] \qquad \text{take } f(z) = z \quad \forall |z| \le 1.$$

$$\varphi_{X_n}(\xi) \longrightarrow \varphi_X(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

$$\therefore X_n \to_w X.$$

Conversely, if $X_n \to_w X$, $\xi \in \mathbb{R}^d$, then for any $u \in \mathbb{R}$,

$$\varphi_{\xi \cdot X_n}(u) = \mathbb{E}[e^{i(u\xi) \cdot X_n}] \to \mathbb{E}[e^{i(u\xi) \cdot X}]$$

$$= \varphi_{\xi \cdot X}(u).$$

$$\xi \cdot X_n \to_w \xi \cdot X.$$

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**Theorem**: (Multivariate CLT) If $\{\mathbb{X}_n\}_{n=1}^{\infty}$ are iid random vectors in $\mathbb{R}^d$ with $L^2$ entries, and $Q = \mathbb{E}[\mathring{\mathbb{X}}_1 \mathring{\mathbb{X}}_1^T]$, then $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathring{\mathbb{X}}_j \to_w Z \overset{d}{=} \mathcal{N}(0,Q)$.

Pf. Fix $\xi \in \mathbb{R}^d$. Let $X_n^{\xi} := \xi \cdot \mathbb{X}_n$. Then $\{X_n^{\xi}\}_{n=1}^{\infty}$ are independent, and

$$\varphi_{X_n^{\xi}}(u) = \mathbb{E}[e^{iu\xi\cdot\mathbb{X}_n}] = \varphi_{\mathbb{X}_n}(u\xi) = \varphi_{\mathbb{X}_1}(u\xi)$$

$\therefore \{X_n^{\xi}\}_{n=1}^{\infty}$ are iid. They are in $L^2$:

$$\mathbb{E}[X_n^{\xi}] = \mathbb{E}[\xi\cdot\mathbb{X}_n] = \xi\cdot\mathbb{E}[\mathbb{X}_n] = \xi\cdot\mathbb{E}[\mathbb{X}_1]$$

$$(v\cdot w)^2 = v^T w\, v^T w = v^T(w w^T)v = v\cdot(ww^T)v$$

$$\begin{aligned}
\mathrm{Var}[X_n^{\xi}] &= \mathbb{E}[(\xi\cdot\mathbb{X}_n)^2] - (\xi\cdot\mathbb{E}[\mathbb{X}_n])^2 \\
&= \mathbb{E}[\xi\cdot\mathbb{X}_n\mathbb{X}_n^T\xi] - \xi\cdot\mathbb{E}[\mathbb{X}_n]\mathbb{E}[\mathbb{X}_n]^T\xi \\
&= \xi\cdot(\underbrace{\mathbb{E}[\mathbb{X}_n\mathbb{X}_n^T] - \mathbb{E}[\mathbb{X}_n]\mathbb{E}[\mathbb{X}_n]^T}_{\mathbb{E}[(\mathbb{X}_n - \mathbb{E}[\mathbb{X}_n])(\mathbb{X}_n - \mathbb{E}[\mathbb{X}_n])^T] = Q})\xi = \xi\cdot Q\xi.
\end{aligned}$$

$\therefore$ By basic CLT,

$$\frac{1}{\sqrt{Q\xi\cdot\xi}}\,\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(X_j^{\xi} - \xi\cdot\mathbb{E}[\mathbb{X}_1]) \to_w \mathcal{N}(0,1)$$

$$\xi\cdot\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathring{\mathbb{X}}_j\Big) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\xi\cdot(\mathbb{X}_j - \mathbb{E}[\mathbb{X}_j]) \to_w \sqrt{Q\xi\cdot\xi}\; Z \overset{d}{=} \xi\cdot Z$$

$\mathcal{N}(0,Q)$

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