If $\mu_n \to_w \mu$, then $\hat{\mu}_n(\xi) \to \hat{\mu}(\xi) \quad \forall \xi \in \mathbb{R}^d$

Amazingly, the converse also holds! Even better:

Theorem: Let $\{\mu_n\}_{n=1}^\infty \subset \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Suppose that $\varphi(\xi) := \lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}^d$.
If φ is continuous @ 0, then $\exists!\, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. $\varphi = \hat{\mu}$, and $\mu_n \to_w \mu$.

Eg. We saw that if $X \stackrel{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, then $\varphi_{(X_1+\cdots+X_n)/\sqrt{n}}(\xi) \to e^{-\xi^2/2}$

$$\Rightarrow \quad \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to_w X \stackrel{d}{=} \mathcal{N}(0,1).$$

Lemma: If $\mu, \nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then

$$\int_{\mathbb{R}^d} \hat{\mu}(\xi)\, \nu(d\xi) = \int_{\mathbb{R}^d} \hat{\nu}(x)\, \mu(dx)$$

Pf.

$$\| $$

$$\int_{\mathbb{R}^d} \nu(d\xi) \int_{\mathbb{R}^d} e^{ix\cdot\xi} \mu(dx)$$

Cor: $\displaystyle \int_{\mathbb{R}^d} [1 - \mathrm{Re}\, \hat{\mu}(\xi)]\, \nu(d\xi) = \int_{\mathbb{R}^d} [1 - \mathrm{Re}\, \hat{\nu}(x)]\, \mu(dx).$

Prop: (Characteristic tail estimate)

Let ρ be a probability density on $\mathbb{R}^d$, supported in $\bar{B}_1$.
Let $M>0$ be such that $|\hat{\rho}(\xi)| \leq \frac{1}{2}$ for all $|\xi| \geq M$.
Then $\forall \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and $a>0$,

$$\mu\{x \in \mathbb{R}^d : |x| \geq a\} \leq 2 \int_{\bar{B}_1} \left[1 - \mathrm{Re}\,\hat{\mu}\left(\tfrac{M}{a} x\right)\right] \rho(x)\, dx.$$

E.g. $(d=1)$ Take $\rho(x) = \frac{1}{2} \mathbb{1}_{|x| \leq 1}$. Then $\hat{\rho}(\xi) = \frac{\sin \xi}{\xi}$, and

$$\therefore \text{ for any r.v. } X, \quad \mathbb{P}(|X| \geq a) \leq \int_{-1}^{1} \left[1 - \mathrm{Re}\,\varphi_X\left(\tfrac{2}{a} x\right)\right] dx.$$

$$= \frac{a}{2} \int_{-2/a}^{2/a} \left[1 - \mathrm{Re}\,\varphi_X(u)\right] du.$$

Pf. Let $\varepsilon > 0$ and set $\nu(dx) = \varepsilon^{-d} \rho(x/\varepsilon)\, dx$.

$$\therefore \hat{\nu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} \varepsilon^{-d} \rho(x/\varepsilon)\, dx$$

$$\therefore \int_{\mathbb{R}^d} [1 - \mathrm{Re}\, \hat{\rho}(\varepsilon\xi)]\, \mu(d\xi)$$

$$= \int_{\mathbb{R}^d} [1 - \mathrm{Re}\, \hat{\mu}(\xi)]\, \varepsilon^{-d} \rho(\xi/\varepsilon)\, d\xi$$

$$= \int_{\mathbb{R}^d} [1 - \mathrm{Re}\, \hat{\mu}(\varepsilon x)]\, \rho(x)\, dx$$

Cor: If $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ are s.t.

$\varphi(\xi) := \lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}^d$ and

φ is continuous @ $\xi = 0$, then

$\{\mu_n\}_{n=1}^{\infty}$ is tight.

Pf. Fix ρ, M as in the tail estimate proposition:

$$\mu_n\{x \in \mathbb{R}^d : |x| \geq a\} \leq 2 \int_{\overline{B}_1} [1 - \mathrm{Re}\, \hat{\mu}_n(\tfrac{M}{a} x)]\, \rho(x)\, dx$$

Theorem: Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Suppose that $\varphi(\xi) := \lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}^d$. If φ is continuous @ 0, then $\exists!\, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. $\varphi = \hat{\mu}$, and $\mu_n \to_w \mu$.

Pf. By the preceding corollary, $\{\mu_n\}_{n=1}^{\infty}$ is tight.
$\therefore$ By Prokhorov, $\exists$ subsequence $\mu_{n_k} \to_w \mu$ for some $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

Claim: $\mu_n \to_w \mu$.

Otherwise: $\exists\, g \in C_b(\mathbb{R}^d)$ s.t. $\int g\, d\mu_n \not\to \int g\, d\mu$