If $\mu_n \to_w \mu$, then $\hat{\mu}_n(\xi) \to \hat{\mu}(\xi) \quad \forall \xi \in \mathbb{R}^d$

$$\begin{array}{ccc} \hat{\mu}_n(\xi) & & \hat{\mu}(\xi) \\ \| & & \| \\ \int e_\xi \, d\mu_n & & \int e_\xi \, d\mu \end{array}$$

$$e_\xi(x) = e^{i\xi \cdot x} \in C_b(\mathbb{R}^d)$$

Amazingly, the converse also holds! Even better:

Theorem: Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Suppose that $\varphi(\xi) := \lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}^d$.
If φ is continuous @ 0, then $\exists!\, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. $\varphi = \hat{\mu}$, and $\mu_n \to_w \mu$.

Eg. We saw that if $X \stackrel{d}{=} \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, then $\varphi_{(X_1 + \cdots + X_n)/\sqrt{n}}(\xi) \to e^{-\xi^2/2}$

$$\Rightarrow \quad \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to_w X \stackrel{d}{=} \mathcal{N}(0,1).$$

Lemma: If $\mu, \nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then

$$\mathrm{Re} \int_{\mathbb{R}^d} \hat{\mu}(\xi)\, \nu(d\xi) = \mathrm{Re} \int_{\mathbb{R}^d} \hat{\nu}(x)\, \mu(dx)$$

Pf.

$$\int_{\mathbb{R}^d} \nu(d\xi) \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, \mu(dx) \overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} \mu(dx) \underbrace{\int_{\mathbb{R}^d} e^{ix\cdot\xi}\, \nu(d\xi)}_{\hat{\nu}(x)}$$

Cor:

$$\int_{\mathbb{R}^d} [1 - \mathrm{Re}\,\hat{\mu}(\xi)]\, \nu(d\xi) = \int_{\mathbb{R}^d} [1 - \mathrm{Re}\,\hat{\nu}(x)]\, \mu(dx).$$

Prop: (Characteristic tail estimate)

Let ρ be a probability density on $\mathbb{R}^d$, supported in $\bar{B}_1$.
Let $M>0$ be such that $|\hat{\rho}(\xi)| \le \frac{1}{2}$ for all $|\xi| \ge M$.
Then $\forall \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and $a>0$,

$$\mu\{x \in \mathbb{R}^d : |x| \ge a\} \le 2\int_{\bar{B}_1} \left[1 - \mathrm{Re}\,\hat{\mu}\left(\tfrac{M}{a}x\right)\right]\rho(x)\,dx.$$

E.g. $(d=1)$ Take $\rho(x) = \frac{1}{2}\mathbb{1}_{|x|\le 1}$. Then $\hat{\rho}(\xi) = \frac{\sin\xi}{\xi}$, and

$$|\hat{\rho}(\xi)| \le \frac{1}{|\xi|} \le \frac{1}{2} \Leftarrow |\xi| \ge 2 =: M.$$

$\therefore$ for any r.v. X, $\mathbb{P}(|X| \ge a) \le \int_{-1}^{1} \left[1 - \mathrm{Re}\,\varphi_X\left(\tfrac{2}{a}x\right)\right]dx$.

$$= \frac{a}{2}\int_{-2/a}^{2/a} \left[1 - \mathrm{Re}\,\varphi_X(u)\right]du.$$

Pf. Let $\varepsilon > 0$ and set $\nu(dx) = \varepsilon^{-d}\rho(x/\varepsilon)\,dx$.

$$\therefore \hat{\nu}(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\,\varepsilon^{-d}\rho(x/\varepsilon)\,dx$$

$$= \int_{\mathbb{R}^d} e^{i\varepsilon\xi\cdot y}\rho(y)\,dy = \hat{\rho}(\varepsilon\xi)$$

$$\therefore \int_{\mathbb{R}^d}[1-\mathrm{Re}\,\hat{\rho}(\varepsilon\xi)]\,\mu(d\xi) = \int_{\mathbb{R}^d}[1-\mathrm{Re}\,\hat{\nu}(\xi)]\,\mu(d\xi)$$

$$= \int_{\mathbb{R}^d}[1-\mathrm{Re}\,\hat{\mu}(\xi)]\,\nu(d\xi)$$

$$= \int_{\mathbb{R}^d}[1-\mathrm{Re}\,\hat{\mu}(\xi)]\,\varepsilon^{-d}\rho(\xi/\varepsilon)\,d\xi$$

$$= \int_{\cancel{\mathbb{R}^d}\ \overline{B}_1}[1-\mathrm{Re}\,\hat{\mu}(\varepsilon x)]\,\underline{\rho(x)\,dx}$$

$\frac{M}{a}\ x$

$$\geq \int_{\mathbb{R}^d}\underbrace{[1-\mathrm{Re}\,\hat{\rho}(\varepsilon\xi)]}_{\geq 1/2}\mathbb{1}_{|\varepsilon\xi|\geq M}\,\mu(d\xi)$$

$$\geq \frac{1}{2}\mu\{\xi\in\mathbb{R}^d : |\varepsilon\xi|\geq M\}$$

$$|\xi| \geq \underbrace{M/\varepsilon}_{a}$$

///

Cor: If $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ are s.t.

$\varphi(\xi) := \lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}^d$ and

φ is continuous @ $\xi = 0$, then

$\{\mu_n\}_{n=1}^{\infty}$ is tight.

Pf. Fix ρ, M as in the tail estimate proposition:

$$\mu_n\{x \in \mathbb{R}^d : |x| \geq a\} \leq 2\int_{\bar{B}_1} [1 - \mathrm{Re}\,\hat{\mu}_n(\tfrac{M}{a}x)]\rho(x)\,dx$$

$(\star)$ diff $< \frac{\varepsilon}{2}$ $\forall$ large n

$$\xrightarrow[n\to\infty]{DCT} 2\int_{\bar{B}_1} \underbrace{[1 - \mathrm{Re}\,\varphi(\tfrac{M}{a}x)]}\rho(x)\,dx \leq 2\delta(a)$$

$$\delta(a) = \sup_{|x|\leq 1} |1 - \mathrm{Re}\,\varphi(\tfrac{M}{a}x)|$$

Since φ is cont. @ 0, $\lim_{a\to\infty} \delta(a) = 0$.

Fix $\varepsilon > 0$, choose a large enough that $\delta(a) < \frac{\varepsilon}{4}$.

Choose N s.t. $n \geq N$ $(\star)$ holds.

$$\mu_n(\mathbb{R}^d \setminus \bar{B}_a) \leq 2\delta(a) + \frac{\varepsilon}{2} < \varepsilon.$$

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Theorem: Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Suppose that $\varphi(\xi) := \lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}^d$.
If φ is continuous @ 0, then $\exists!\ \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ s.t. $\varphi = \hat{\mu}$, and $\mu_n \to_w \mu$.

Pf. By the preceding corollary, $\{\mu_n\}_{n=1}^{\infty}$ is tight.
$\therefore$ By Prokhorov, $\exists$ subsequence $\mu_{n_k} \to_w \mu$ for some $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

$$\hat{\mu}_{n_k}(\xi) \to \hat{\mu}(\xi) \quad \forall \xi \in \mathbb{R}^d$$
$$\hat{\mu}_{n_k}(\xi) \to \varphi(\xi), \quad \hat{\mu}(\xi) = \varphi(\xi)$$

Claim: $\mu_n \to_w \mu$.

Otherwise: $\exists\ g \in C_b(\mathbb{R}^d)$ s.t. $\int g\,d\mu_n \not\to \int g\,d\mu$

i.e. $\exists\ \varepsilon > 0$, $\exists\ n_k'$ s.t. $\left| \int g\,d\mu_{n_k'} - \int g\,d\mu \right| \geq \varepsilon \quad \forall k$

By Prokhorov, $\exists$ further subsequence $\{n_k''\} \subseteq \{n_k'\}$
s.t. $\mu_{n_k''} \to_w \nu$ for some $\nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

$$\hat{\mu}_{n_k''} \to \hat{\nu}, \quad \hat{\mu}_{n_k''} \to \hat{\mu}, \quad \hat{\nu} = \hat{\mu} \Rightarrow \nu = \mu. \quad \therefore \mu_{n_k''} \to_w \mu.$$

$$\therefore \cancel{\left| \int g\,d\mu_{n_k''} - \int g\,d\mu \right| \to 0} \quad \left(\geq \varepsilon\right)$$