Using Dynkin's complex multiplicative systems theorem, we saw that

$$\mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \ni \mu \longmapsto \hat{\mu}$$

is an injective map. So, in principal, we can recover μ from $\hat{\mu}$. Also in practice

Theorem: If $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then for $a<b$ in $\mathbb{R}$,

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{\mu}(\xi) \left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi} \right) d\xi = \mu((a,b))$$

$\xi \longmapsto \dfrac{e^{-ia\xi} - e^{-ib\xi}}{\xi}$ is bounded. But $\hat{\mu}$ may not be in L^1

Lemma: For $r \geq 0$, define $S(r) = \int_{-r}^{r} \frac{\sin \xi}{\xi}\, d\xi$.

Then S is continuous on $[0,\infty)$, and

$$\lim_{r\to\infty} S(r) = \pi.$$

Pf. "One simple trick": $\frac{1}{\xi} = \int_0^\infty e^{-\xi t}\,dt$.

$$\therefore \int_{-r}^{r} \frac{\sin\xi}{\xi}\, d\xi = 2\int_0^r \frac{\sin\xi}{\xi}\, d\xi = 2\int_0^r \sin\xi \int_0^\infty e^{-\xi t}\,dt\, d\xi$$

Pf. (Fourier Inversion Formula)

$$I(R) := \frac{1}{2\pi}\int_{-R}^{R} \hat{\mu}(\xi)\left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}\right) d\xi$$

$$= \frac{1}{2\pi}\int_{-R}^{R}\int_{\mathbb{R}} e^{i\xi x}\mu(dx)\left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}\right) d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\int_{-R}^{R}\left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}\right) d\xi \quad \mu(dx)$$

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{-R}^{R}\underbrace{\left(\frac{e^{-i(a-x)\xi} - e^{-i(b-x)\xi}}{i\xi}\right)} d\xi \; \mu(dx)$$

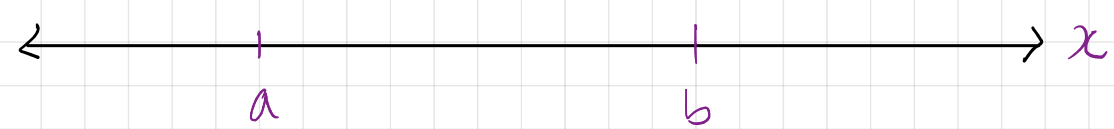
$$\frac{e^{-i\alpha\xi}}{i\xi} = -\frac{i}{\xi}\cos(\alpha\xi) - \frac{1}{\xi}\sin(\alpha\xi)$$

$$\therefore \quad I(R) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{-R}^{R} \frac{\sin(\xi(x-a)) - \sin(\xi(x-b))}{\xi} \, d\xi \; \mu(dx)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}}$$

$$\therefore \quad \lim_{R \to \infty} I(R) = \frac{1}{2} \int_{\mathbb{R}} [\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b)] \; \mu(dx)$$

$\operatorname{sgn}(x-a)$



$\operatorname{sgn}(x-b)$

$\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b)$

Cor: If $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\hat{\mu} \in L^1(\lambda)$,

then $\mu \ll \lambda$ and its density $\rho = \frac{d\mu}{d\lambda}$

is $\rho(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(\xi)\, e^{-i\xi x}\, d\xi$

Pf. Define ρ by $\uparrow$; so $\rho \in C_0(\mathbb{R})$

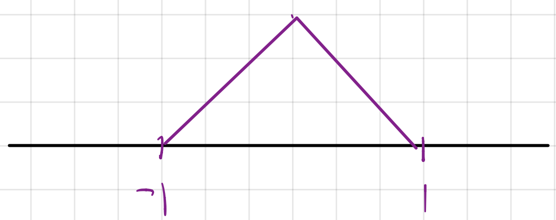
$$\int_a^b \rho(x)\,dx = \frac{1}{2\pi} \int_a^b dx \int_{\mathbb{R}} \hat{\mu}(\xi)\, e^{-i\xi x}\, d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(\xi)\, d\xi \int_a^b e^{-i\xi x}\, dx$$

$$= \frac{1}{2\pi} \int \hat{\mu}(\xi)\, d\xi \left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi} \right) = \mu(a,b) + \tfrac{1}{2}\mu\{a,b\}.$$

Eg. If $X \overset{d}{=} N(0,1)$, $\varphi_X(\xi) = e^{-\xi^2/2}$.

Eg. $\rho(x) = (1-|x|)_+$



$$\hat{\rho}(\xi) = \int_{-1}^{1} (1-|x|)\, e^{ix\xi}\, dx = \int_{0}^{1} (1-x)\, e^{ix\xi}\, dx + \int_{-1}^{0} (1+x)\, e^{ix\xi}\, dx$$

$$= 2\,\mathrm{Re} \int_0^1 (1-x)\, e^{ix\xi}\, dx$$

$$= 2\,\mathrm{Re}\left(\int_0^1 e^{ix\xi}\, dx + i \frac{d}{d\xi} \int_0^1 e^{ix\xi}\, dx \right)$$

$$= 2\,\mathrm{Re}\left[\left(1 + i \frac{\partial}{\partial \xi}\right) \left(\frac{e^{i\xi} - 1}{i\xi} \right) \right]$$

$$= 2\, \frac{1-\cos\xi}{\xi^2}.$$