Using Dynkin's complex multiplicative systems theorem, we saw that

$$\mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \ni \mu \longmapsto \hat{\mu}$$

is an injective map. So, in principal, we can recover μ from $\hat{\mu}$. Also in practice (at least for $d=1$)

Theorem: If $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then for $a<b$ in $\mathbb{R}$,

$$\lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \hat{\mu}(\xi) \left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi} \right) d\xi = \mu((a,b)) + \frac{1}{2}\mu\{a,b\}$$

& $$\mu\{a\} = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} e^{-ia\xi} \hat{\mu}(\xi)\, d\xi$$

$\xi \mapsto \left| \dfrac{e^{-ia\xi} - e^{-ib\xi}}{\xi} \right|$ is bounded. But $\hat{\mu}$ may not be in L^1

$$= \left| -i \int_a^b e^{-i\xi x} dx \right| \leq \int_a^b dx = |b-a|$$

Lemma: For $r \geq 0$, define $S(r) = \int_{-r}^{r} \frac{\sin\xi}{\xi}\, d\xi$.

Then S is continuous on $[0,\infty)$, and

$$\lim_{r\to\infty} S(r) = \pi.$$

Pf. "One simple trick": $\frac{1}{\xi} = \int_0^\infty e^{-\xi t}\, dt$.

$$\therefore \int_{-r}^{r} \underbrace{\frac{\sin\xi}{\xi}}_{\text{even}}\, d\xi = 2\int_0^r \frac{\sin\xi}{\xi}\, d\xi = 2\int_0^r \sin\xi \int_0^\infty e^{-\xi t}\, dt\, d\xi$$

$$\overset{\text{Fubini}}{=} 2\int_0^\infty \underbrace{\int_0^r e^{-\xi t}\sin\xi\, d\xi}_{=\frac{1}{1+t^2}\left[1 - e^{-rt}(\cos r + t\sin r)\right]}\, dt$$

$$= \pi - 2\int_0^\infty \frac{1}{1+t^2} e^{-rt}(\cos r + t\sin r)\, dt \xrightarrow{DCT} 0 \text{ as } r\to\infty$$

$\therefore S(r) \in C^\infty$

Pf. (Fourier Inversion Formula)

$$I(R) := \frac{1}{2\pi}\int_{-R}^{R} \hat{\mu}(\xi)\left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}\right)d\xi$$

$$= \frac{1}{2\pi}\int_{-R}^{R}\int_{\mathbb{R}} e^{i\xi x}\mu(dx)\left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}\right)d\xi$$

Fubini

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\int_{-R}^{R}\left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi}\right)d\xi \;\mu(dx)$$

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{-R}^{R}\underbrace{\left(\frac{e^{-i(a-x)\xi} - e^{-i(b-x)\xi}}{i\xi}\right)}\,d\xi\;\mu(dx)$$

$$\frac{e^{-i\alpha\xi}}{i\xi} = -\frac{i}{\xi}\cos(\alpha\xi) - \frac{1}{\xi}\sin(\alpha\xi)$$

odd

$$\therefore \quad I(R) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-R}^{R} \underbrace{\frac{\sin(\xi(x-a)) - \sin(\xi(x-b))}{\xi}}\, d\xi \;\; \mu(dx)$$

Change of variables $\eta = \xi(x-a)$

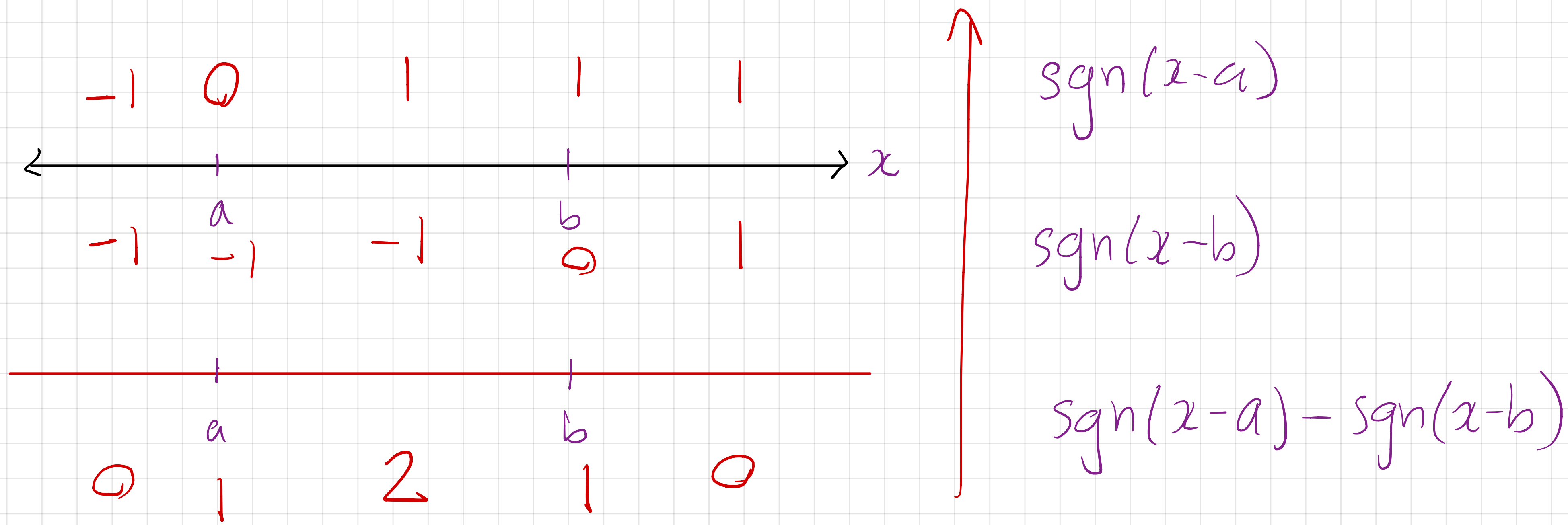
$$\int_{-R}^{R} \frac{\sin(\xi(x-a))}{\xi}\, d\xi = \operatorname{sgn}(x-a)\, S(R|x-a|)$$

$$\operatorname{sgn}(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \Big[\operatorname{sgn}(x-a)\, \underbrace{S(R|x-a|)}_{\to \pi} - \operatorname{sgn}(x-b)\, \underbrace{S(R|x-b|)}_{\to \pi}\Big]\mu(dx)$$

as $R \to \infty$

$$\therefore \quad \lim_{R\to\infty} I(R) = \frac{1}{2}\int_{\mathbb{R}} [\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b)]\, \mu(dx) = \int \left(\mathbb{1}_{(a,b)} + \frac{1}{2}\mathbb{1}_{\{a,b\}}\right) d\mu$$



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Cor: If $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\hat{\mu} \in L^1(\lambda)$,

then $\mu \ll \lambda$ and its density $\rho = \frac{d\mu}{d\lambda}$

is $\rho(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(\xi)\, e^{-i\xi x}\, d\xi = \frac{1}{2\pi} (\hat{\mu})^\wedge(-x)$

Pf. Define ρ by $\uparrow$; so $\rho \in C_0(\mathbb{R})$

$$\int_a^b \rho(x)\,dx = \frac{1}{2\pi} \int_a^b dx \int_{\mathbb{R}} \hat{\mu}(\xi)\, e^{-i\xi x}\, d\xi$$

Fubini

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\mu}(\xi)\, d\xi \int_a^b e^{-i\xi x}\, dx$$

$$\frac{e^{-\xi b}}{-i\xi} - \frac{e^{-i\xi a}}{-i\xi}$$

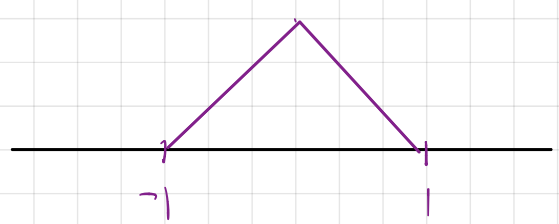
$$= \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R}^{R} \hat{\mu}(\xi)\, d\xi \left(\frac{e^{-ia\xi} - e^{-ib\xi}}{i\xi} \right) = \mu(a,b) + \cancel{\tfrac{1}{2}\mu\{a,b\}}.$$

Take $a \uparrow b$ ///

Eg. If $X \overset{d}{=} N(0,1)$, $\varphi_X(\xi) = e^{-\xi^2/2}$. $\leftarrow \in L^1(\mathbb{R}, \lambda))$

$\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \nearrow$ $\quad f_X(x) = \frac{1}{2\pi}\hat{\varphi}_X(-x) = \frac{1}{\sqrt{2\pi}}\hat{\gamma}(-x) = \frac{1}{\sqrt{2\pi}} e^{-(-x)^2/2} = \gamma(x)$.

Eg. $\rho(x) = (1-|x|)_+$



$$\hat\rho(\xi) = \int_{-1}^{1} (1-|x|)\,e^{ix\xi}\,dx = \int_0^1 (1-x)\,e^{ix\xi}\,dx + \int_{-1}^{0} (1+x)\,e^{ix\xi}\,dx$$

$$= 2\,\mathrm{Re}\int_0^1 (1-x)\,e^{ix\xi}\,dx$$

$$= 2\,\mathrm{Re}\left(\int_0^1 e^{ix\xi}\,dx + i\frac{d}{d\xi}\int_0^1 e^{ix\xi}\,dx\right)$$

$$= 2\,\mathrm{Re}\left[\left(1 + i\frac{\partial}{\partial\xi}\right)\left(\frac{e^{i\xi}-1}{i\xi}\right)\right]$$

$$= 2\,\frac{1-\cos\xi}{\xi^2}.$$

$\therefore \quad \frac{1}{\pi}\int_{\mathbb{R}} \frac{1-\cos\xi}{\xi^2}\, e^{-ix\xi}\, d\xi = (1-|x|)_+$.

$\Rightarrow \quad \frac{1}{\pi}\frac{1-\cos x}{x^2}$ is a Prob density,

$\wedge \quad = \quad (1-|-\xi|)_+ = (1-|\xi|)_+$