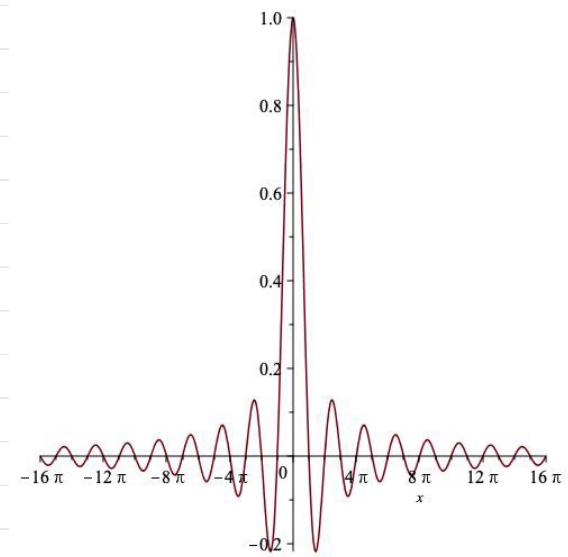
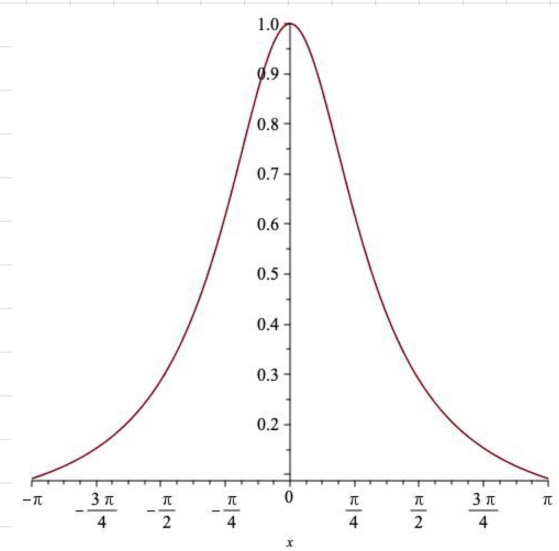
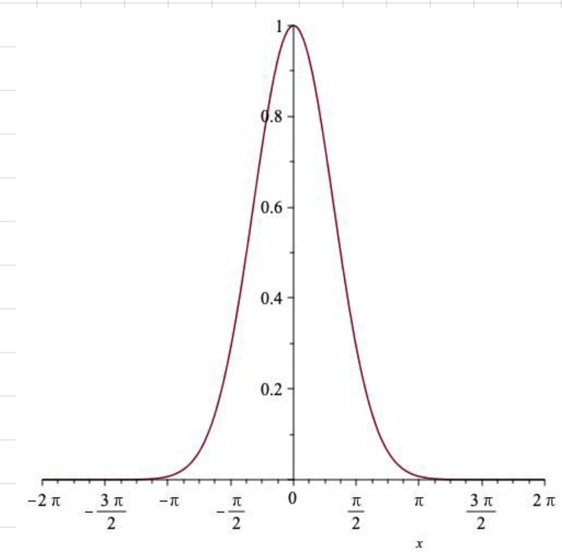
We saw several examples of characteristic functions last time:



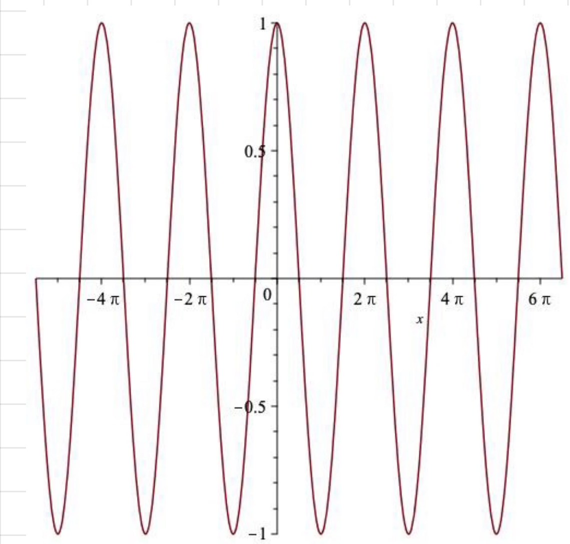
$\dfrac{\sin\xi}{\xi}$



$\dfrac{1}{1+\xi^2}$



$e^{-\xi^2/2}$



$\cos\xi$

All are continuous (in fact C^∞). But the last one is different:

If $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ has a density $d\mu = \rho\, d\lambda$ wrt Lebegue measure, we denote $\hat{\mu} = \hat{\rho}$. I.e. if $\rho \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$,

$$\hat{\rho}(\xi) := \int e^{i\xi \cdot x} \rho(x)\, dx .$$

Lemma: (Riemann-Lebesgue) If $\rho \in L^1$, $\hat{\rho} \in C_0$; i.e. $\hat{\rho}(\xi) \to 0$ as $|\xi| \to \infty$.

Pf. Step 1: Assume $\rho \in C_c^\infty(\mathbb{R}^d)$. Then for $1 \le j \le d$,

$$i\xi_j \hat{\rho}(\xi) = \int_{\mathbb{R}^d} \rho(x)\, i\xi_j\, e^{i\xi \cdot x}\, dx$$

$\therefore \; |\xi_j| \, |\hat{\rho}(\xi)| \le$

Step 2: For general $\rho \in L^1(\mathbb{R}^d, \lambda)$,
approximate by C_c^∞ functions. [Driver, Thm 17.29]

- $\rho \mathbb{1}_{|\rho| \leq M} \to \rho$ in L^1 as $M \to \infty$

  Thus, we may assume ρ is bounded.

- $\rho \mathbb{1}_{\bar{B}_R} \to \rho$ in L^1 as $R \to \infty$

  Thus, we may assume $\operatorname{supp} \rho \subseteq \bar{B}_R$.

- Let $\mathbb{H} = \{ h \in B(\bar{B}_R) \text{ s.t. } \exists \{\varphi_n\} \text{ in } C_c^\infty(\bar{B}_R) \text{ with } \|h - \varphi_n\|_{L^1} \to 0 \}$

  ↳ $1 \in \mathbb{H}$

  ↳ closed under bounded convergence.

- Let $\mathbb{M} = C_c^\infty(\bar{B}_R)$.

Step 3: Combine. Let $\varepsilon > 0$, and $\psi \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$\|\rho - \psi\|_{L^1} < \frac{\varepsilon}{2}.$$

Then $\forall \xi \in \mathbb{R}^d$ $|\hat{\rho}(\xi) - \hat{\psi}(\xi)|$

In Step 1, we showed $\hat{\psi}(\xi) = O(1/|\xi|)$.

Thus, choose R s.t.