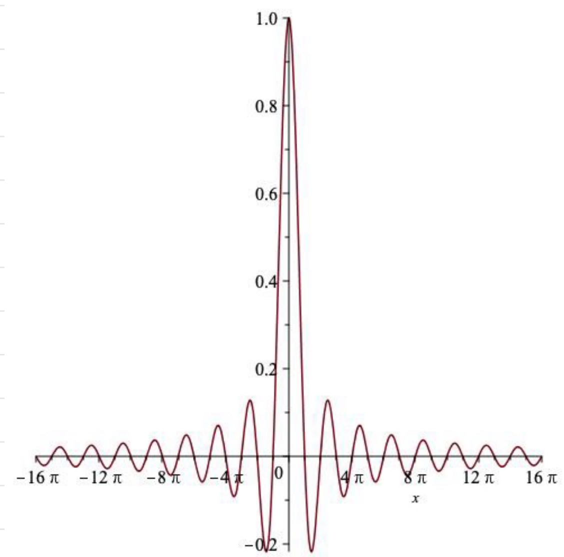
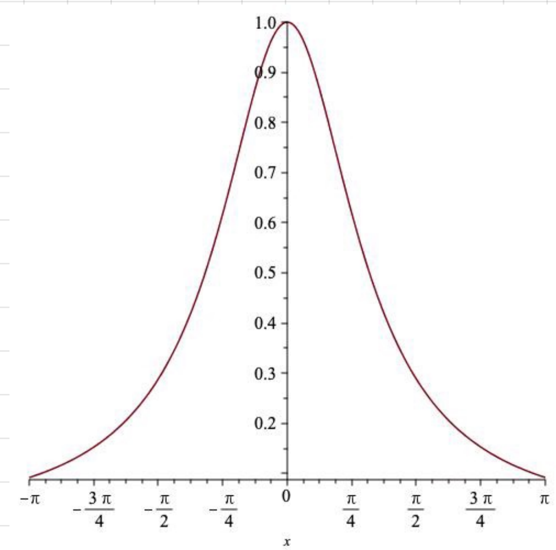
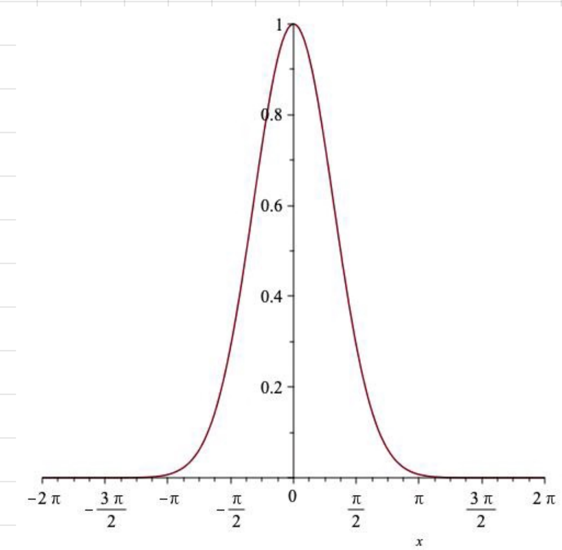
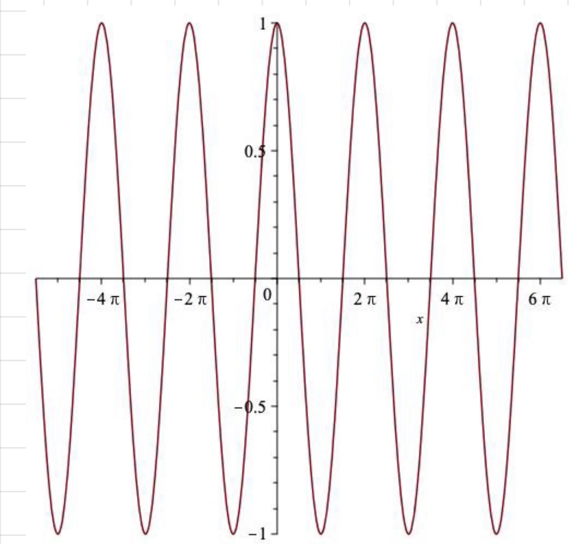
We saw several examples of characteristic functions last time:

$\dfrac{\sin\xi}{\xi}$ $\qquad$ $\dfrac{1}{1+\xi^2}$ $\qquad$ $e^{-\xi^2/2}$ $\qquad$ $\cos\xi$

$\widehat{\mathrm{Unif}[-1,1]}$ $\qquad$ $\widehat{\mathrm{Exp}_{\pm}(1)}$ $\qquad$ $\widehat{\mathcal{N}(0,1)}$ $\qquad$ $\widehat{\tfrac{1}{2}(\delta_1+\delta_{-1})}$

All are continuous (in fact C^∞). But the last one is different:

C_0 $\quad \lim_{|\xi|\to\infty}\varphi(\xi)=0$ $\qquad\qquad$ $\notin C_0$

a.c. measures

If $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ has a density $d\mu = \rho\, d\lambda$ wrt Lebegue measure, we denote $\hat{\mu} = \hat{\rho}$. I.e. if $\rho \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$,

$$\hat{\rho}(\xi) := \int e^{i\xi\cdot x} \rho(x)\, dx.$$

Lemma: (Riemann-Lebesgue) If $\rho \in L^1$, $\hat{\rho} \in C_0$; i.e. $\hat{\rho}(\xi) \to 0$ as $|\xi| \to \infty$.

Pf. Step 1: Assume $\rho \in C_c^\infty(\mathbb{R}^d)$. Then for $1 \le j \le d$,

$$i\xi_j \hat{\rho}(\xi) = \int_{\mathbb{R}^d} \rho(x) \underbrace{i\xi_j e^{i\xi\cdot x}}_{\frac{\partial}{\partial x_j} e^{i\xi\cdot x}} dx \underset{\text{integration by parts}}{=} -\int_{\mathbb{R}^d} \frac{\partial \rho}{\partial x_j}(x) e^{i\xi\cdot x} dx$$

$$\therefore \quad |\xi_j|\,|\hat{\rho}(\xi)| \le \int_{\mathbb{R}^d} \left| \frac{\partial \rho}{\partial x_j}(x) \right| dx = M_j < \infty.$$

C_c^∞,

$$\left.|\xi|\,|\hat{\rho}(\xi)| \le \sqrt{M_1^2 + \cdots + M_d^2} := M < \infty \right\} \quad |\hat{\rho}(\xi)| \le \frac{M}{|\xi|} \to 0$$

Step 2: For general $\rho \in L^1(\mathbb{R}^d, \lambda)$,

approximate by C_c^∞ functions. [Driver, Thm 17.29]

- $\rho \mathbb{1}_{|\rho| \le M} \to \rho$ in L^1 as $M \to \infty$ by DCT

Thus, we may assume ρ is bounded.

- $\rho \mathbb{1}_{\bar{B}_R} \to \rho$ in L^1 as $R \to \infty$ by DCT

Thus, we may assume $\operatorname{supp} \rho \subseteq \bar{B}_R$.

- Let $\mathbb{H} = \{ h \in \mathcal{B}(\bar{B}_R) \text{ s.t. } \exists \{\varphi_n\} \text{ in } C_c^\infty(\bar{B}_R) \text{ with } \|h - \varphi_n\|_{L^1} \to 0 \}$

$\hookrightarrow$ $1 \in \mathbb{H}$ $\quad \varphi_n \in C_c^\infty \quad \varphi_n = 1$ on $\bar{B}_{R - 1/n}$

$\hookrightarrow$ closed under bounded convergence. ✓ $\quad \therefore h \in \mathbb{H}$

$\mathbb{H} \ni h_n \to h$, $|h_n| \le M$ $\quad \therefore \int_{\bar{B}_R} |h_n - h| \, d\lambda \to 0$ by DCT. $\Uparrow$

$\hookrightarrow$ $\exists \varphi_n \in C_c^\infty$ s.t. $\|h_n - \varphi_n\| \le \frac{1}{n}$. $\quad \|h - \varphi_n\|_{L^1} \le \underbrace{\|h - h_n\|_{L^1}}_{\searrow 0} + \underbrace{\|h_n - \varphi_n\|_{L^1}}_{\searrow 0}$

- Let $\mathbb{M} = C_c^\infty(\bar{B}_R)$. $\leftarrow$ mult. systems.

$\mathbb{M} \subseteq \mathbb{H}$

$\therefore$ by Dynkin $\quad \mathcal{B}(\bar{B}_R, \sigma(\mathbb{M})) \subseteq \mathbb{H}$. $\quad$ use e_ξ φ $\nwarrow$ bump supp in $\bar{B}_R$

$= \mathcal{B}(\bar{B}_R)$

Step 3: Combine. Let $\varepsilon > 0$, and $\psi \in C_c^\infty(\mathbb{R}^d)$ s.t.

$$\|\rho - \psi\|_{L^1} < \frac{\varepsilon}{2}.$$

Then $\forall \xi \in \mathbb{R}^d$ $\quad |\hat{\rho}(\xi) - \hat{\psi}(\xi)| = \left| \int (\rho(x) - \psi(x))\, e^{i\xi \cdot x}\, dx \right|$

$$\leq \int |\rho(x) - \psi(x)|\, dx < \frac{\varepsilon}{2}.$$

In Step 1, we showed $\quad \hat{\psi}(\xi) = O(1/|\xi|)$.

Thus, choose R s.t. $\quad |\xi| \geq R \Rightarrow |\hat{\psi}(\xi)| < \varepsilon/2$

$$\Downarrow$$

$$|\hat{\rho}(\xi)| = |\hat{\rho}(\xi) - \hat{\psi}(\xi) + \hat{\psi}(\xi)|$$

$$\leq |\hat{\rho}(\xi) - \hat{\psi}(\xi)| + |\hat{\psi}(\xi)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon,$$

$\Rightarrow \quad |\hat{\rho}(\xi)| \to 0$ as $|\xi| \to \infty$.

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