Characteristic Functions

Def: Let $\mu \in \text{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
For $\xi \in \mathbb{R}^d$, define

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x}\,\mu(dx)$$

the Fourier transform of μ.

If $\underline{X} : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a random vector, its characteristic function $\varphi_{\underline{X}} : \mathbb{R}^d \to \mathbb{C}$ is

$$\varphi_{\underline{X}}(\xi) = \hat{\mu}_{\underline{X}}(\xi)$$

Prop: $\mu \mapsto \hat{\mu}$ is injective: if $\hat{\mu}(\xi) = \hat{\nu}(\xi)$ $\forall \xi \in \mathbb{R}^d$, then $\mu = \nu$.

Pf. The final Corollary of [Lecture 24.1]:
If $\int e_\xi \, d\mu = \int e_\xi \, d\nu$ $\forall \xi \in \mathbb{R}^d$, then $\mu = \nu$

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Thus, in principle, we can recover μ from $\hat{\mu}$.
I.e. μ_X is determined by φ_X (and vice versa).

/ Compare: the moment generating function over $\mathbb{R}$:

$$M_X(t) = \mathbb{E}[e^{tX}], \quad \varphi_X(\xi) = \mathbb{E}[e^{i\xi X}]$$

requires exponential integrability to be defined — always defined, and carries at least as much info. /

Prop: (Basic Properties of the Fourier Transform $\hat{\mu}$ of $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$)

1. $\hat{\mu}(0) = 1$ and $|\hat{\mu}(\xi)| \le 1 \quad \forall \xi \in \mathbb{R}^d$.
2. $\underline{\hat{\mu} \in C_{\mathbb{C}}(\mathbb{R}^d)}$
3. $\overline{\hat{\mu}(\xi)} = \hat{\mu}(-\xi) \quad \forall \xi \in \mathbb{R}^d$. In particular, $\hat{\mu}$ is $\mathbb{R}$-valued iff μ is symmetric ($\mu(B) = \mu(-B) \ \forall B \in \mathcal{B}(\mathbb{R}^d)$) (if $\mu = \mu_X$, $X \overset{d}{=} -X$)
4. If $\int_{\mathbb{R}^d} |x|^k \mu(dx) < \infty$ then $\hat{\mu} \in C^k_{\mathbb{C}}$ and $\frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} \hat{\mu}(\xi) = \int_{\mathbb{R}^d} (ix_{j_1}) \cdots (ix_{j_k}) e^{i\xi \cdot x} \mu(dx)$

Pf. 1. $\hat{\mu}(0) = \int e^{i0\cdot x}\mu(dx)$

$|\hat{\mu}(\xi)| = \left|\int e^{i\xi\cdot x}\mu(dx)\right|$

2. If $\xi_n \to \xi$ in $\mathbb{R}^d$, then $e_{\xi_n}(x) = e^{i\xi_n\cdot x}$

3. $\overline{\hat{\mu}(\xi)} = \overline{\int e^{i\xi\cdot x}\mu(dx)}$

$\therefore$ If μ is symmetric, let $X \overset{d}{=} \mu$

Conversely, let $\nu(B) := \mu(-B)$. If $\hat{\mu}(\xi) = \hat{\mu}(-\xi)$

4. $\frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} e^{i\xi \cdot x} = (ix_{j_1}) \cdots (ix_{j_k}) e^{i\xi\cdot x} \quad \forall x \in \mathbb{R}^d$.

[Lecture 10.1] Differentiate under the $\int$

- need $\left| \frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} e^{i\xi\cdot x} \right| \leq g_K(x)$ for some $g_K \in L^1(\mu)$, $\forall \xi \in K$

Prop: If $\mu, \nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then

$$\widehat{\mu * \nu} = \hat{\mu} \cdot \hat{\nu}$$

I.e. If X, Y are independent random vectors in $\mathbb{R}^d$, then

$$\varphi_{X+Y}(\xi) = \varphi_X(\xi) \cdot \varphi_Y(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

Moreover: if $a \in \mathbb{R}$, $v \in \mathbb{R}^d$, then $\varphi_{aX+v}(\xi) = e^{i\xi \cdot v} \varphi_X(a\xi)$.

Pf. $\varphi_{X+Y}(\xi) = \mathbb{E}\left[e^{i\xi \cdot (X+Y)}\right]$

E.g. $U \overset{d}{=} \mathrm{Unif}([a,b])$

$$\varphi_U(\xi) = \int_a^b \frac{1}{b-a} e^{i\xi x}\, dx = \frac{e^{ib\xi} - e^{ia\xi}}{i(b-a)\xi}$$

Special case: if $a = -b$, $\varphi_U(\xi) = \dfrac{e^{ib\xi} - e^{-ib\xi}}{2ib\xi}$

E.g. $N \overset{d}{=} \mathrm{Poisson}(\lambda)$

$$\varphi_N(\xi) = \mathbb{E}[e^{i\xi N}] = \sum_{n=0}^{\infty} e^{i\xi n} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$

E.g. $T \overset{d}{=} \mathrm{Exp}(\lambda)$

$$\varphi_T(\xi) = \int_0^\infty \lambda e^{-\lambda x} e^{ix\xi}\, dx$$

Variant: bilateral exponential $T_\pm$ with density $f(x) = \frac{\lambda}{2} e^{-\lambda|x|}$.

$$\varphi_{T_\pm}(\xi) = \int_{\mathbb{R}} \frac{\lambda}{2} e^{-\lambda|x|} e^{ix\xi}\, dx$$

Eg. $Y \stackrel{d}{=}$ Rademacher : $\mathbb{P}(Y = \pm 1) = \frac{1}{2}$

$\varphi_X(\xi) = \mathbb{E}[e^{i\xi Y}] =$

So, if $Y_1, \dots, Y_n$ are iid Rademachers, $S_n = Y_1 + \cdots + Y_n$

$\varphi_{S_n}(\xi) =$

Rescale : $\varphi_{S_n/b_n}(\xi) =$

Eg. $X \overset{d}{=} \mathcal{N}(0,1)$.

$$\varphi_X(\xi) = \mathbb{E}[e^{i\xi X}] = \int_{\mathbb{R}} e^{i\xi x} (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

$$\varphi_X'(\xi) = \mathbb{E}[iXe^{i\xi X}]$$