Characteristic Functions

Def: Let $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
For $\xi \in \mathbb{R}^d$, define

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{i\xi \cdot x}\,\mu(dx) = \int e_\xi \, d\mu .$$

the Fourier transform of μ.

If $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a random vector, its characteristic function $\varphi_X : \mathbb{R}^d \to \mathbb{C}$ is

$$\varphi_X(\xi) = \hat{\mu}_X(\xi) = \mathbb{E}[e^{i\xi \cdot X}]$$

Prop: $\mu \mapsto \hat{\mu}$ is injective: if $\hat{\mu}(\xi) = \hat{\nu}(\xi)$ $\forall \xi \in \mathbb{R}^d$, then $\mu = \nu$.

Pf. The final Corollary of [Lecture 24.1]:
If $\underbrace{\int e_\xi \, d\mu}_{\hat{\mu}(\xi)} = \underbrace{\int e_\xi \, d\nu}_{\hat{\nu}(\xi)}$ $\forall \xi \in \mathbb{R}^d$, then $\mu = \nu$

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Thus, in principle, we can recover μ from $\hat{\mu}$.
I.e. μ_X is determined by φ_X (and vice versa).

/ Compare: the moment generating function over $\mathbb{R}$:

$$M_X(t) = \mathbb{E}[e^{tX}], \quad \varphi_X(\xi) = \mathbb{E}[e^{i\xi X}] = M_X(i\xi)$$

requires exponential integrability to be defined

always defined, and carries at least as much info. /

Prop: (Basic Properties of the Fourier Transform $\hat{\mu}$ of $\mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$)

1. $\hat{\mu}(0) = 1$ and $|\hat{\mu}(\xi)| \leq 1 \quad \forall \xi \in \mathbb{R}^d$.
2. $\hat{\mu} \in C_{\mathbb{C}}(\mathbb{R}^d)$
3. $\overline{\hat{\mu}(\xi)} = \hat{\mu}(-\xi) \quad \forall \xi \in \mathbb{R}^d$. In particular, $\hat{\mu}$ is $\mathbb{R}$-valued iff μ is symmetric ($\mu(B) = \mu(-B) \ \forall B \in \mathcal{B}(\mathbb{R}^d)$) (if $\mu = \mu_X$, $X \overset{d}{=} -X$)
4. If $\int_{\mathbb{R}^d} |x|^k \mu(dx) < \infty$ then $\hat{\mu} \in C^k_{\mathbb{C}}$ and $\frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} \hat{\mu}(\xi) = \int_{\mathbb{R}^d} (ix_{j_1}) \cdots (ix_{j_k}) e^{i\xi \cdot x} \mu(dx)$

Pf. 1. $\hat{\mu}(0) = \int e^{i \cancel{0} \cdot x}_{1} \mu(dx) = 1$.

$$|\hat{\mu}(\xi)| = \left| \int e^{i\xi \cdot x} \mu(dx) \right| \le \int \underbrace{|e^{i\xi \cdot x}|}_{1} \mu(dx) = 1.$$

2. If $\xi_n \to \xi$ in $\mathbb{R}^d$, then $e_{\xi_n}(x) = e^{i\xi_n \cdot x} \to e^{i\xi \cdot x} = e_\xi(x)$ $\forall x \in \mathbb{R}^d$

$$\hat{\mu}(\xi_n) = \int e_{\xi_n} d\mu \to \int e_\xi d\mu = \hat{\mu}(\xi).$$

3. $\overline{\hat{\mu}(\xi)} = \overline{\int e^{i\xi \cdot x} \mu(dx)} = \int \overline{e^{i\xi \cdot x}} \mu(dx) = \int e^{-i\xi \cdot x} \mu(dx) = \hat{\mu}(-\xi)$.

$\therefore$ If μ is symmetric, let $X \overset{d}{=} \mu \overset{d}{=} -X$.

$$\overline{\hat{\mu}(\xi)} = \hat{\mu}(-\xi) = \varphi_X(-\xi) = E[e^{-i\xi \cdot X}] = E[e^{i\xi \cdot X}] = \hat{\mu}(\xi)$$

$\therefore \hat{\mu}(\mathbb{R}^d) \subseteq \mathbb{R}$.

Conversely, let $\nu(B) := \mu(-B)$. If $\hat{\mu}(\xi) = \hat{\mu}(-\xi) = \hat{\nu}(\xi)$ $\forall \xi \in \mathbb{R}^d$. $\Rightarrow \mu = \nu$.

4. $\displaystyle\int \frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} e^{i\xi\cdot x} \, d\mu = \int (i x_{j_1}) \cdots (i x_{j_k}) e^{i\xi\cdot x} \, d\mu \quad \forall x \in \mathbb{R}^d$.

$$\overset{?}{\parallel}$$

$$\frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} \int e^{i\xi\cdot x} \, d\mu = \frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} \hat{\mu}(\xi).$$

[Lecture 10.1] Differentiate under the $\int$

- need $\displaystyle\underbrace{\left| \frac{\partial}{\partial \xi_{j_1}} \cdots \frac{\partial}{\partial \xi_{j_k}} e^{i\xi\cdot x} \right|}$ $\le g_K(x)$ for some $g_K \in L^1(\mu)$, $\forall \xi \in K$

$= |x_{j_1} \cdots x_{j_k}|$ indep. of ξ

$\in L^1(\mu)$ by assumption.

$\therefore \infty > \int |x|^k \, d\mu$ $\qquad |x| = \sqrt{x_1^2 + \cdots + x_d^2}$

$\ge \int |x_{j_1}| \cdots |x_{j_k}| \, d\mu$ $\qquad \ge |x_j| \quad \forall j$.

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Prop: If $\mu, \nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then

$$\widehat{\mu * \nu} = \hat{\mu} \cdot \hat{\nu}$$

I.e. If X, Y are independent random vectors in $\mathbb{R}^d$, then

$$\varphi_{X+Y}(\xi) = \varphi_X(\xi) \cdot \varphi_Y(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

Moreover: if $a \in \mathbb{R}$, $v \in \mathbb{R}^d$, then $\varphi_{aX+v}(\xi) = e^{i\xi \cdot v} \varphi_X(a\xi)$.

Pf. $\varphi_{X+Y}(\xi) = \mathbb{E}[\underbrace{e^{i\xi \cdot (X+Y)}}_{e^{i\xi \cdot X}\, e^{i\xi \cdot Y}}] = \mathbb{E}[e^{i\xi \cdot X}]\, \mathbb{E}[e^{i\xi \cdot Y}] = \varphi_X(\xi)\, \varphi_Y(\xi)$.

indep.

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E.g. $U \overset{d}{=} \mathrm{Unif}([a,b])$

$$\varphi_U(\xi) = \int_a^b \frac{1}{b-a} e^{i\xi x}\, dx = \frac{e^{ib\xi} - e^{ia\xi}}{i(b-a)\xi}$$

Special case: if $a=-b$, $\varphi_U(\xi) = \frac{e^{ib\xi} - e^{-ib\xi}}{2ib\xi} = \frac{\sin(b\xi)}{b\xi}$ "sinc function"

E.g. $N \overset{d}{=} \mathrm{Poisson}(\lambda)$

$$\varphi_N(\xi) = \mathbb{E}[e^{i\xi N}] = \sum_{n=0}^{\infty} e^{i\xi n} \cdot e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^{i\xi}\lambda)^n}{n!} = e^{-\lambda} e^{\lambda e^{i\xi}}$$

$$= e^{\lambda(e^{i\xi}-1)}$$

E.g. $T \overset{d}{=} \mathrm{Exp}(\lambda)$

$$\varphi_T(\xi) = \int_0^\infty \lambda e^{-\lambda x} \underbrace{e^{ix\xi}}_{\cos(x\xi) + i\sin(x\xi)} dx = \lambda \int_0^\infty e^{-(\lambda - i\xi)x} dx = \frac{\lambda}{\lambda - i\xi} \quad \text{valid } \forall \xi \in \mathbb{R}$$

$$M_T(t) = \frac{\lambda}{\lambda - t} \quad t < \lambda.$$

Variant: bilateral exponential $T_\pm$ with density $f(x) = \frac{\lambda}{2} e^{-\lambda|x|}$.

$$\varphi_{T_\pm}(\xi) = \int_{\mathbb{R}} \frac{\lambda}{2} e^{-\lambda|x|} e^{ix\xi} dx = \int_0^\infty + \int_{-\infty}^0 = \frac{1}{2}\left(\frac{\lambda}{\lambda - i\xi} + \frac{\lambda}{\lambda + i\xi}\right) = \frac{\lambda^2}{\lambda^2 + \xi^2}$$

Eg. $Y \overset{d}{=}$ Rademacher: $\mathbb{P}(Y=\pm 1) = \frac{1}{2}$ $\quad \mathbb{E}[Y] = \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$. $\mathrm{Var}\,Y = 1$.

$$\varphi_X(\xi) = \mathbb{E}[e^{i\xi Y}] = \frac{1}{2}e^{i\xi 1} + \frac{1}{2}e^{i\xi(-1)} = \cos\xi.$$

So, if $Y_1, \dots, Y_n$ are iid Rademachers, $S_n = Y_1 + \dots + Y_n$

$$\varphi_{S_n}(\xi) = \varphi_{Y_1}(\xi) \cdots \varphi_{Y_n}(\xi) = (\cos\xi)^n$$

Rescale: $\varphi_{S_n/b_n}(\xi) = (\cos(\xi/b_n))^n = (1 + \frac{1}{2}(\xi/b_n)^2 + \dots)^n$

$= 1 + \dots$

$$\log \varphi_{S_n/b_n}(\xi) = n \log\cos(\xi/b_n) = n \cdot (-\sec^2(\eta/b_n))\xi^2/b_n^2$$

η between $0, \xi$

Taylor expansion:

$|t| < \frac{\pi}{2}$. $\rightarrow \log\cos t = \cancel{\log\cos 0} + \cancel{\log\cos' (0)\, t} + \frac{1}{2}\log\cos''(\eta)\, t^2$

for some η between $0, t$.

$\log\cos' = -\tan$

$\log\cos'' = -\sec^2$

$b_n = n$: $\log \varphi_{\frac{S_n}{n}}(\xi) \to 0$.

Take $b_n = \sqrt{n}$. $\quad \log\varphi_{S_n/\sqrt{n}}(\xi) \to -\frac{1}{2}\xi^2$.

$\varphi_{S_n/\sqrt{n}}(\xi) \to e^{-\frac{1}{2}\xi^2}$.

Eg. $X \stackrel{d}{=} \mathcal{N}(0,1)$.

$$\varphi_X(\xi) = \mathbb{E}[e^{i\xi X}] = \int_{\mathbb{R}} e^{i\xi x} (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

has finite moments of all orders. $\therefore \varphi_X \in C^\infty$

$$\varphi_X'(\xi) = \mathbb{E}[iXe^{i\xi X}] = -\xi\, \mathbb{E}[e^{i\xi X}] = -\xi \varphi_X(\xi) \Rightarrow \varphi_X(\xi) = \underset{\substack{\| \\ 1}}{\varphi_X(0)}\, e^{-\xi^2/2}.$$

$$= \mathbb{E}[Xf(X)] = \mathbb{E}[f'(X)]$$

$$f(x) = i e^{i\xi x}$$

$$f'(x) = i \cdot i\xi e^{i\xi x} = -\xi e^{i\xi x}$$

If $X_1, \dots, X_n$ i.i.d. Rademachers,

$$\varphi\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) \longrightarrow \varphi_{\mathcal{N}(0,1)}.$$