Integration of $\mathbb{C}$-Valued Functions

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.
Let $f : \Omega \to \mathbb{C}$ be Borel measurable

Def: $f \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$ iff

In this case, define $\int_\Omega f \, d\mu :=$

Note: $\max\{|u|, |v|\} \leq \sqrt{u^2+v^2}$

$\therefore f \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$

$\|f\|_{L^1_{\mathbb{C}}} :=$

It is routine to verify that $\int_\Omega \cdot \, d\mu$ is linear on $L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$, and all the other basic integration properties hold: once we verify the correct triangle inequality.

Prop: If $f \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$, then $\left| \int_\Omega f \, d\mu \right| \leq \int_\Omega |f| \, d\mu$.

Pf. $\int f \, d\mu = re^{i\theta}$, where $r =$.

$$r = e^{-i\theta} \int f \, d\mu$$

Any result about $\int_\Omega \, d\mu$ involving $|f|$ $\therefore$ extends to $\mathbb{C}$-valued.

Dynkin Revisited

Recall Dynkin's Multiplicative Systems Theorem: [Lec 14.1]

Let $\mathbb{H} \subseteq \mathbb{B}(\Omega)$ be a subspace, containing 1, and closed under bounded convergence

Let $\mathbb{M} \subseteq \mathbb{H}$ be a multiplicative system

Then $\mathbb{H}$ contains all bounded $\sigma(\mathbb{M})$-measurable functions:

$$\mathbb{B}(\Omega, \sigma(\mathbb{M})) \subseteq \mathbb{H}.$$

Pf. $\mathbb{M}_0 := \mathrm{span}_{\mathbb{C}}(\mathbb{M} \cup \{1\})$

$\sigma(\mathbb{M}) = \sigma(\mathbb{M}_0)$.

$\mathbb{H}^{\mathbb{R}} := \{ f \in \mathbb{H} : \mathrm{Im} f \subseteq \mathbb{R} \}$

$\mathbb{M}_0^{\mathbb{R}} := \{ f \in \mathbb{M}_0 : \mathrm{Im} f \subseteq \mathbb{R} \}$

Cor: Suppose $\mathbb{H}$ is a $\mathbb{C}$-space of bounded Borel functions on $\mathbb{R}^d$ that is closed under $\mathbb{C}$-conjugation and bounded convergence.

For $\xi \in \mathbb{R}^d$, set $e_\xi(x) = e^{i\xi\cdot x}$.

If $e_\xi \in \mathbb{H}$ $\forall \xi \in \mathbb{R}^d$, then $B_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \subseteq \mathbb{H}$.

Pf. $1 = e_0$, so $1 \in \mathbb{M} := \{e_\xi : \xi \in \mathbb{R}^d\} \subseteq \mathbb{H}$.

Also $\overline{e_\xi} =$

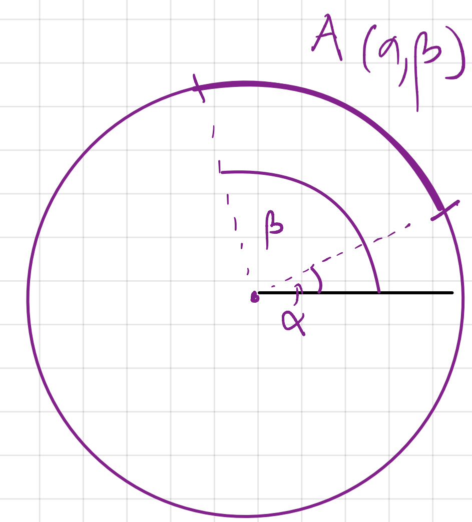
and $e_\xi \cdot e_\eta(x) =$

Claim: if $\mathbb{M} = \{ e_\xi : \xi \in \mathbb{R} \}$ then $\sigma(\mathbb{M}) = \mathcal{B}(\mathbb{R})$.

- $e_\xi \in C(\mathbb{R})$

- If $\xi > 0$, $e_\xi^{-1}(A(\alpha,\beta))$

$$= \bigsqcup_{n \in \mathbb{Z}} \left(\left(\frac{\alpha}{\xi}, \frac{\beta}{\xi} \right) + 2\pi \frac{n}{\xi} \right)$$



Cor: Let $\mu, \nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Note that $e_\xi \in \mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \subseteq L^1(\mu), L^1(\nu)$.

Suppose that $\int_{\mathbb{R}^d} e_\xi \, d\mu = \int_{\mathbb{R}^d} e_\xi \, d\nu \quad \forall \xi \in \mathbb{R}$.

Then $\mu = \nu$.

Pf. Let $\mathbb{H} = \{ f \in \mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) : \int_{\mathbb{R}^d} f \, d\mu = \int_{\mathbb{R}^d} f \, d\nu \}$.

- Closed under $\mathbb{C}$-conjugation
- Closed under bounded convergence
- $e_\xi \in \mathbb{H} \;\; \forall \xi \in \mathbb{R}^d$ by assumption.

$\therefore$ By preceding corollary, $\mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \subseteq \mathbb{H}$