Integration of $\mathbb{C}$-Valued Functions

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

Let $f : \Omega \to \mathbb{C}$ be Borel measurable (wrt $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}^2)$)

$f(\omega) = u(\omega) + i v(\omega)$, $u, v : \Omega \to \mathbb{R}$ $\Updownarrow$ u, v $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable.

Def: $f \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$ iff $u, v \in L^1(\Omega, \mathcal{F}, \mu)$

In this case, define $\int_\Omega f \, d\mu := \int_\Omega u \, d\mu + i \int_\Omega v \, d\mu$ $\qquad \overline{\int f \, d\mu} = \int \bar{f} \, d\mu$.

Note: $\max\{|u|, |v|\} \le \underbrace{\sqrt{u^2 + v^2}}_{=|f|} \le \sqrt{2} \max\{|u|, |v|\}$

$\therefore f \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$ iff $|f| \in L^1(\Omega, \mathcal{F}, \mu)$ i.e. $\int_\Omega |f| \, d\mu < \infty$

$\|f\|_{L^1_{\mathbb{C}}} := \int_\Omega |f| \, d\mu$

It is routine to verify that $\int_\Omega \cdot \, d\mu$ is linear on $L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$, and all the other basic integration properties hold: once we verify the correct triangle inequality.

Prop: If $f \in L^1_{\mathbb{C}}(\Omega, \mathcal{F}, \mu)$, then $\left|\int_\Omega f\, d\mu\right| \leq \int_\Omega |f|\, d\mu$.

Pf. $\int f\, d\mu = re^{i\theta}$, where $r = \left|\int_\Omega f\, d\mu\right|$.

$$\mathbb{R} \ni r = e^{-i\theta}\int f\, d\mu = \int e^{-i\theta} f\, d\mu = \int \underbrace{\mathrm{Re}(e^{-i\theta} f)}_{\leq |\mathrm{Re}(e^{-i\theta}f)|}\, d\mu + \underbrace{\cancel{i\int \mathrm{Im}(e^{-i\theta} f)\, d\mu}}_{0}$$

$$r \leq \int \underbrace{|\mathrm{Re}(e^{-i\theta} f)|}_{\leq \sqrt{|\mathrm{Re}(e^{-i\theta}f)|^2 + |\mathrm{Im}(e^{-i\theta}f)|^2} = |e^{-i\theta} f| = |f|}\, d\mu \leq \int |f|\, d\mu.$$

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Any result about $\int_\Omega \, d\mu$ involving $|f|$ $\therefore$ extends to $\mathbb{C}$-valued. Eg. DCT.

Dynkin Revisited

Recall Dynkin's Multiplicative Systems Theorem: [Lec 14.1]

Let $\mathbb{H} \subseteq \mathbb{B}_{\mathbb{C}}(\Omega)$ be a $\mathbb{C}$-subspace, containing $\mathbb{1}$, and closed under bounded convergence

Let $\mathbb{M} \subseteq \mathbb{H}$ be a multiplicative system

} closed under $\mathbb{C}$-conjugation

Then $\mathbb{H}$ contains all bounded $\mathbb{C}$-valued $\sigma(\mathbb{M})$-measurable functions:

$$\mathbb{B}_{\mathbb{C}}(\Omega, \sigma(\mathbb{M})) \subseteq \mathbb{H}.$$

$$f = \frac{f+\bar{f}}{2} + i\,\frac{f-\bar{f}}{2i} \rightsquigarrow \begin{cases} \mathbb{M}_0 = \mathbb{M}_0^{\mathbb{R}} + i\,\mathbb{M}_0^{\mathbb{R}} \\ \mathbb{H} = \mathbb{H}^{\mathbb{R}} + i\,\mathbb{H}^{\mathbb{R}} \end{cases}$$

Pf. $\mathbb{M}_0 := \mathrm{span}_{\mathbb{C}}(\mathbb{M} \cup \{1\})$ is a $\mathbb{C}$-algebra, $\mathbb{M}_0 \subseteq \mathbb{H}$, $\mathbb{M}_0$ closed under $\mathbb{C}$-conj.

$\sigma(\mathbb{M}) = \sigma(\mathbb{M}_0) = \sigma(\mathbb{M}_0^{\mathbb{R}})$

$\mathbb{H}^{\mathbb{R}} := \{ f \in \mathbb{H} : \mathrm{Im} f \subseteq \mathbb{R} \}$ $\quad 1 \in \mathbb{H}^{\mathbb{R}}$, $\mathbb{R}$-subspace, closed under bdd conv.

$\mathbb{M}_0^{\mathbb{R}} := \{ f \in \mathbb{M}_0 : \mathrm{Im} f \subseteq \mathbb{R} \}$ $\quad \mathbb{M}_0^{\mathbb{R}}$ is a mult. system, $\mathbb{M}_0^{\mathbb{R}} \subseteq \mathbb{H}^{\mathbb{R}}$

By Dynkin, $\mathbb{B}_{\mathbb{R}}(\Omega, \cancel{\sigma(\mathbb{M}_0^{\mathbb{R}})}\ \sigma(\mathbb{M})) \subseteq \mathbb{H}^{\mathbb{R}}$

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Cor: Suppose $\mathbb{H}$ is a $\mathbb{C}$-space of bounded Borel functions on $\mathbb{R}^d$ that is closed under $\mathbb{C}$-conjugation and bounded convergence.

For $\xi \in \mathbb{R}^d$, set $e_\xi(x) = e^{i\xi\cdot x}$.

If $e_\xi \in \mathbb{H}$ $\forall \xi \in \mathbb{R}^d$, then $\mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \subseteq \mathbb{H}$.

Pf. $1 = e_0$, so $1 \in \mathbb{M} := \{e_\xi : \xi \in \mathbb{R}^d\} \subseteq \mathbb{H}$.

Also $\overline{e_\xi}(x) = \overline{e^{i\xi\cdot x}} = e^{-i\xi\cdot x} = e_{-\xi}(x)$ $\therefore$ $\overline{\mathbb{M}} \subseteq \mathbb{M}$.

and $e_\xi \cdot e_\eta(x) = e^{i\xi\cdot x}\cdot e^{i\eta\cdot x} = e^{i(\xi+\eta)\cdot x} = e_{\xi+\eta}(x) \in \mathbb{M}$.

$$\therefore \quad \mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \sigma(\mathbb{M})) \subseteq \mathbb{H}.$$

WTS $= \mathcal{B}(\mathbb{R}^d)$

we'll do $d=1$;

general case in [Driver, Cor 12.13]

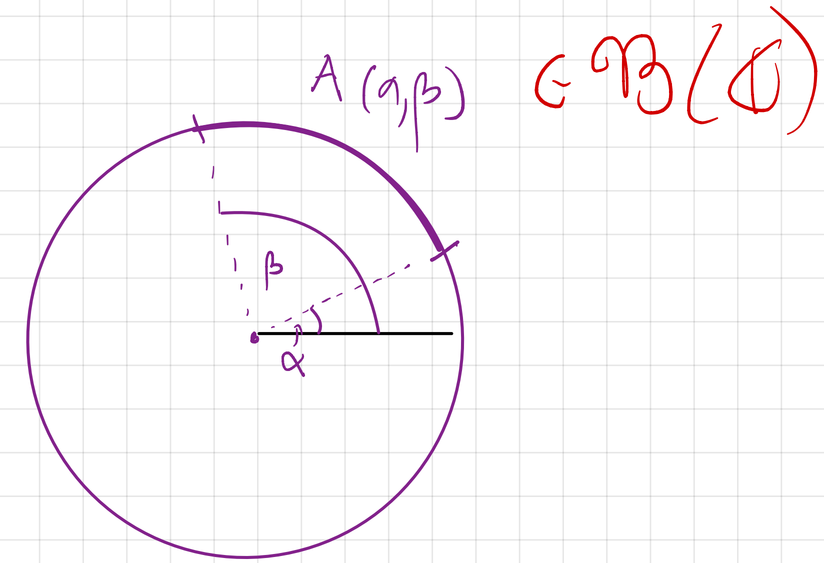
Claim: if $\mathbb{M} = \{ e_\xi : \xi \in \mathbb{R} \}$ then $\sigma(\mathbb{M}) = \mathcal{B}(\mathbb{R})$.

$$e_\xi(x) = e^{i\xi \cdot x}$$

- $e_\xi \in C(\mathbb{R})$ $\therefore$ Borel-meas. $\sigma(\mathbb{M}) \subseteq \mathcal{B}(\mathbb{R})$.

- If $\xi > 0$, $e_\xi^{-1}(A(\alpha,\beta))$

$$= \bigsqcup_{n \in \mathbb{Z}} \left(\left(\frac{\alpha}{\xi}, \frac{\beta}{\xi}\right) + 2\pi \frac{n}{\xi} \right)$$



Choose $a<b$ s.t. $-\pi < \xi a < \xi b < \pi$

$$e_\xi^{-1}(A(\xi a, \xi b)) = \bigsqcup_{n \in \mathbb{Z}} \left((a,b) + 2\pi \frac{n}{\xi} \right)$$

$$\therefore (a,b) = \bigcap_{k \in \mathbb{N}} e_{1/k}^{-1}(A(a/k, b/k)) \in \sigma(\mathbb{M}).$$

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Cor: Let $\mu, \nu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Note that $e_\xi \in \mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \subseteq L^1(\mu), L^1(\nu)$.

Suppose that $\int_{\mathbb{R}^d} e_\xi \, d\mu = \int_{\mathbb{R}^d} e_\xi \, d\nu \quad \forall \xi \in \mathbb{R}$.

Then $\mu = \nu$.

Pf. Let $\mathbb{H} = \{ f \in \mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) : \int_{\mathbb{R}^d} f \, d\underline{\mu} = \int_{\mathbb{R}^d} f \, d\underline{\nu} \}$.

- Closed under $\mathbb{C}$-conjugation $\quad \int \bar{f} \, d\mu = \overline{\int f \, d\mu} = \overline{\int f \, d\nu} = \int \bar{f} \, d\nu$
- Closed under bounded convergence $\quad$ by DCT.
- $e_\xi \in \mathbb{H} \quad \forall \xi \in \mathbb{R}^d$ by assumption.

$\therefore$ By preceding corollary, $\mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \subseteq \mathbb{H}$

$$\underset{\mathbb{1}_B}{\mathbb{B}_{\mathbb{C}}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))} \ni \mathbb{1}_B \quad \forall B \in \mathcal{B}(\mathbb{R}^d)$$

$$\therefore \underbrace{\int \mathbb{1}_B \, d\mu}_{\mu(B)} = \underbrace{\int \mathbb{1}_B \, d\nu}_{\nu(B)}.$$

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