We've seen that weak convergence is weaker than a.s. convergence, L^p convergence, convergence in probability. There is a connection to a.s. convergence that is useful in many contexts, however.

Theorem: (Skorohod)
Let S be a separable metric space, and $\mu_n, \mu \in \mathrm{Prob}(S, \mathcal{B}(S))$. If $\mu_n \to_w \mu$, then there exists a probability space $(\Omega, \mathcal{F}, P)$ and random variables $Y_n, Y : (\Omega, \mathcal{F}) \to (S, \mathcal{B}(S))$ with $Y_n^* P = \mu_{Y_n} = \mu_n$, $Y^* \mathbb{P} = \mu_Y = \mu$, and $Y_n \to Y$ a.s.

(The proof is quite involved. The probability space can be chosen to be $\Omega = (0,1) \times S^{\mathbb{N}}$, $\mathcal{F} = \mathcal{B}((0,1) \times S^{\mathbb{N}})$, and $\mathbb{P}$ a well-chosen infinite product measure (that introduces lots of complicated correlations between the Y_n s).

We'll prove Skorohod's Theorem in the case $S = \mathbb{R}$.

"Inverting" the CDF

Suppose $F : \mathbb{R} \to [0,1]$ is a CDF

If F is strictly increasing,
meaning $\mu(a,b] > 0 \ \forall a<b$, then F is an invertible function,
$F^{-1} : (0,1) \to \mathbb{R}$

Equip $\mathbb{R}$ with Lebsegue measure λ; then F^{-1} becomes a random variable.

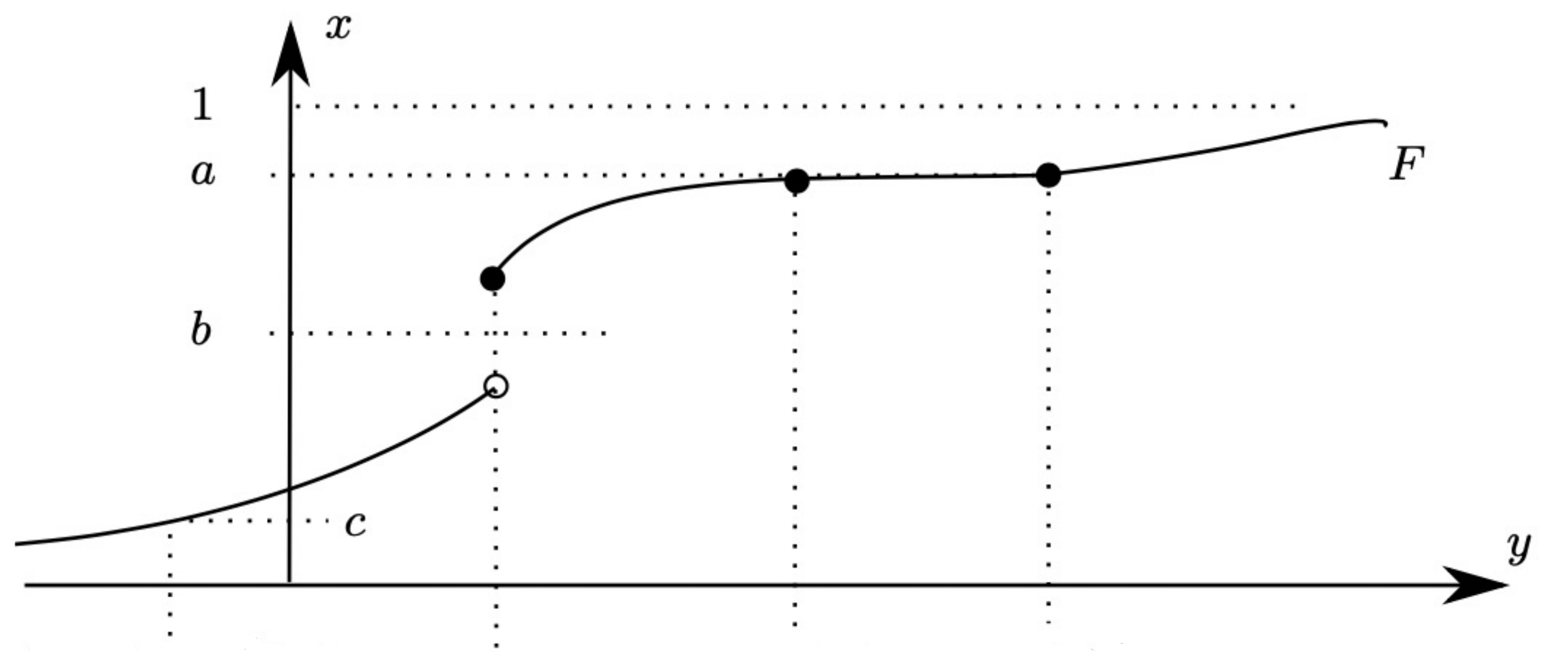
Thus, $\mu_Y =$

It turns out we can make this work even if F is not strictly increasing.

Def Let $F: \mathbb{R} \to [0,1]$ be a CDF. Define

$F^{\leftarrow}: (0,1) \to \mathbb{R}$ $\quad F^{\leftarrow}(x) := \sup\{y \in \mathbb{R} : F(y) < x\}$

$F^{\rightarrow}: (0,1) \to \mathbb{R}$ $\quad F^{\rightarrow}(x) := \inf\{y \in \mathbb{R} : F(y) > x\}$



1. $F^{\leftarrow}(x) \leq F^{\rightarrow}(x)$
   $<$ iff

2. $E = \{x \in (0,1) : F^{\leftarrow}(x) < F^{\rightarrow}(x)\}$ is countable.

3. $\lim\limits_{y \uparrow F^{\leftarrow}(x)} F(y) \leq x \leq F(F^{\leftarrow}(x)) \quad \forall x \in (0,1)$

Lemma: Let $\mu \in \text{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with CDF $F_\mu = F$.
Define $Y : (0,1) \to \mathbb{R}$ to be $Y = F^{\leftarrow}$

$$Y(x) = \sup\{y \in \mathbb{R} : F(y) < x\}$$

Then $Y : ((0,1), \mathcal{B}(0,1)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, and wrt Lebesgue measure λ on $\mathbb{R}$,

$$Y^* \lambda = \mu_Y = \mu.$$

Pf. For $t \in \mathbb{R}$, suppose $Y(x) \leq t$. Then $x \leq F(Y(x))$

In fact: $\{x \in (0,1) : Y(x) \leq t\} =$

Cor: If $\mu \in \text{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $F_\mu = F$, and $U \stackrel{d}{=} \text{Unif}([0,1])$, then $F^{\leftarrow}(U) \stackrel{d}{=} \mu$.

Baby Skorohod Theorem

Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mu_n \to_w \mu$.
Let Y_n, Y be the random variables

$$Y_n = F_{\mu_n}^{\leftarrow}, \quad Y = F_{\mu}^{\leftarrow}$$

on $((0,1), \mathcal{B}(0,1), \lambda)$. Then $Y_n \stackrel{d}{=} \mu_n$, $Y \stackrel{d}{=} \mu$, and $Y_n \to Y$ a.s.

Pf. To complete the proof, we will show that $Y_n(x) \to Y(x)$ for $x \notin E = \{t \in (0,1) : F_\mu^{\leftarrow}(t) < F_\mu^{\rightarrow}(t)\}$

- If $y \in \mathrm{Cont}(F_\mu)$ with $y < Y(x)$

- If $y \in \mathrm{Cont}(F_\mu)$ with $y > Y(x) = F_\mu^{\leftarrow}(x) = F_\mu^{\rightarrow}(x)$

Cor: (Continuous mapping theorem)

Let $f: \mathbb{R} \to \mathbb{R}$ be Borel measurable.

Let $X_n \to_w X$, and suppose $\mathbb{P}(X \in \mathrm{Disc}(f)) = 0$.

Then $f(X_n) \to_w f(X)$. If in addition f is bounded, then $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$.

Pf. Replace X_n, X with Y_n, Y on a probability space where $Y_n \stackrel{d}{=} X_n$, $Y \stackrel{d}{=} X$, and $Y_n \to Y$ a.s.

Let $g \in C_b(\mathbb{R})$; then