We've seen that weak convergence is weaker than a.s. convergence, L^p convergence, convergence in probability. There is a connection to a.s. convergence that is useful in many contexts, however.

Theorem: (Skorohod)
Let S be a separable metric space, and $\mu_n, \mu \in \mathrm{Prob}(S, \mathcal{B}(S))$. If $\mu_n \to_w \mu$, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $Y_n, Y : (\Omega, \mathcal{F}) \to (S, \mathcal{B}(S))$ with $Y_n^* \mathbb{P} = \mu_{Y_n} = \mu_n$, $Y^* \mathbb{P} = \mu_Y = \mu$, and $Y_n \to Y$ a.s.

(The proof is quite involved. The probability space can be chosen to be $\Omega = (0,1) \times S^{\mathbb{N}}$, $\mathcal{F} = \mathcal{B}((0,1) \times S^{\mathbb{N}})$, and $\mathbb{P}$ a well-chosen infinite product measure (that introduces lots of complicated correlations between the Y_n s).

We'll prove Skorohod's Theorem in the case $S = \mathbb{R}$.

"Inverting" the CDF

Suppose $F : \mathbb{R} \to [0,1]$ is a CDF $\quad F = F_\mu$ for some $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

If F is strictly increasing,
meaning $\mu(a,b] > 0 \ \forall a < b$, then F is an invertible function,

$$Y := F^{-1} : (0,1) \to \mathbb{R}$$

Equip $\mathbb{R}$ with Lebesgue measure λ; then F^{-1} becomes a random variable.

$$F_Y(t) = \mathbb{P}(Y \leq t) = \lambda\left(Y^{-1}(-\infty, t]\right) = \lambda\left(F(-\infty, t]\right) = \lambda\left([0, F(t)]\right)$$
$$= F(t)$$

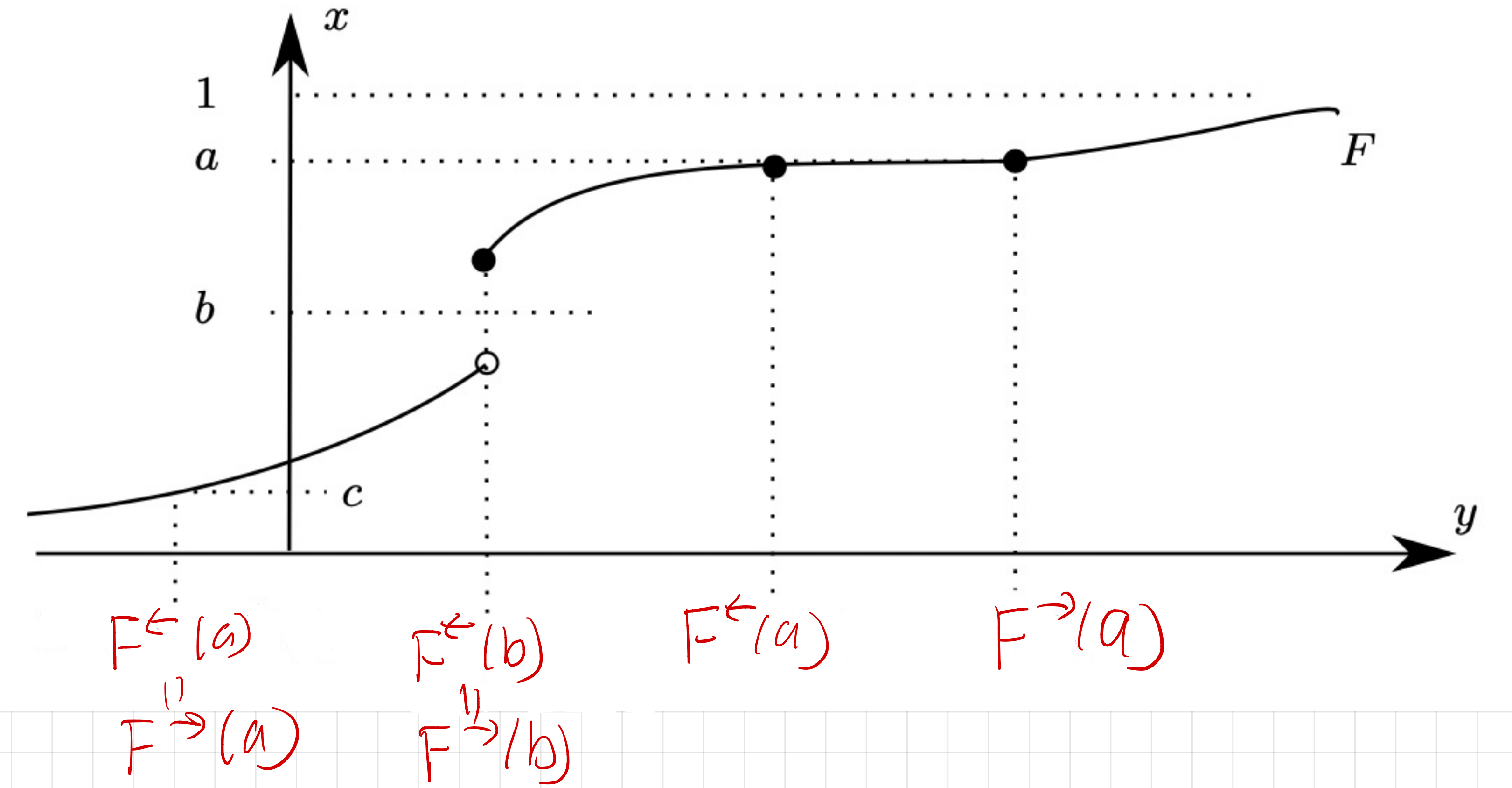
Thus, $\mu_Y = \mu$.

It turns out we can make this work even if F is not strictly increasing.

Def Let $F:\mathbb{R}\to[0,1]$ be a CDF. Define

$F^{\leftarrow}:(0,1)\to\mathbb{R}$ $\quad F^{\leftarrow}(x):=\sup\{y\in\mathbb{R}: F(y)<x\}$

$F^{\rightarrow}:(0,1)\to\mathbb{R}$ $\quad F^{\rightarrow}(x):=\inf\{y\in\mathbb{R}: F(y)>x\}$



3. If $y>F^{\leftarrow}(x)$ then $F(y)\geq x$.
$\therefore$ By right continuity of F, $y\downarrow F^{\leftarrow}(x)$, $F(F^{\leftarrow}(x))\geq x$.
If $y<F^{\leftarrow}(x)$ then $F(y)<x$.
Take $y\uparrow F^{\leftarrow}(x)$

1. $F^{\leftarrow}(x)\leq F^{\rightarrow}(x)$
   $<$ iff x is the height of a flat spot.
2. $E=\{x\in(0,1): F^{\leftarrow}(x)<F^{\rightarrow}(x)\}$ is countable. $\quad \{(F^{\leftarrow}(x),F^{\rightarrow}(x)) : x\in E\}$
   $\ni q\in\mathbb{Q}$
3. $\lim\limits_{y\uparrow F^{\leftarrow}(x)} F(y)\leq x\leq F(F^{\leftarrow}(x)) \quad \forall x\in(0,1)$

Lemma: Let $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with CDF $F_\mu = F$.
Define $Y : (0,1) \to \mathbb{R}$ to be $Y = F^{\leftarrow}$

$$Y(x) = \sup\{ y \in \mathbb{R} : F(y) < x \}$$

Then $Y : ((0,1), \mathcal{B}(0,1)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable,
and wrt Lebesgue measure λ on $\mathbb{R}$,

$$Y^* \lambda = \mu_Y = \mu .$$

Pf. For $t \in \mathbb{R}$, suppose $Y(x) \le t$. Then $x \le F(Y(x)) \le t$.

In fact: $\{ x \in (0,1) : \underline{Y(x) \le t} \} = (0,1) \cap (0, F(t)]$ $\quad \subseteq$

$\supseteq$ If $x \le F(t)$, then $Y(x) = \sup\{ y \in \mathbb{R} : F(y) < x \} \le t$.

$\lambda\{Y \le t\} = \lambda_{(0,1)}((0, F(t)]) = F(t) = F_\mu(t)$. ///

Cor: If $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $F_\mu = F$, and $U \overset{d}{=} \mathrm{Unif}([0,1])$,
then $F^{\leftarrow}(U) \overset{d}{=} \mu$.

Baby Skorohod Theorem

Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mu_n \to_w \mu$.

Let Y_n, Y be the random variables

$$Y_n = F_{\mu_n}^{\leftarrow}, \quad Y = F_{\mu}^{\leftarrow}$$

on $((0,1), \mathcal{B}(0,1), \lambda)$. Then $\underbrace{Y_n \overset{d}{=} \mu_n, \ Y \overset{d}{=} \mu}_{\text{Lemma.}}$, and $Y_n \to Y$ a.s.

Pf. To complete the proof, we will show that $Y_n(x) \to Y(x)$ for $x \notin E = \{t \in (0,1) : F_\mu^{\leftarrow}(t) < F_\mu^{\rightarrow}(t)\}$ is countable $\therefore \lambda(E) = 0$.

- If $y \in \mathrm{Cont}(F_\mu)$ with $y < Y(x) = \sup\{y' : F_\mu(y') < x\}$ $\quad \forall y < Y(x)$

  $\lim_{n\to\infty} F_{\mu_n}(y) = F_\mu(y) < x \quad \therefore F_{\mu_n}(y) < x \ \forall$ large n.

  $Y_n(x) = \sup\{y : F_{\mu_n}(y) < x\} \geq y$

  $\Rightarrow \liminf_{n\to\infty} Y_n(x) \geq Y(x)$

- If $y \in \mathrm{Cont}(F_\mu)$ with $y > Y(x) = F_\mu^{\leftarrow}(x) = F_\mu^{\rightarrow}(x)$ $\rightsquigarrow \therefore \limsup_{n\to\infty} Y_n(x) \leq Y(x)$

///

Cor: (Continuous mapping theorem)
Let $f: \mathbb{R} \to \mathbb{R}$ be Borel measurable.
Let $X_n \to_w X$, and suppose $\mathbb{P}(X \in \mathrm{Disc}(f)) = 0$.
Then $f(X_n) \to_w f(X)$. If in addition f is bounded,
then $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$.

Pf. Replace X_n, X with Y_n, Y on a probability space
where $Y_n \overset{d}{=} X_n$, $Y \overset{d}{=} X$, and $Y_n \to Y$ a.s. (by Skorohod)
Let $g \in C_b(\mathbb{R})$; then

$$\mathrm{Disc}(g \circ f) \subseteq \mathrm{Disc}(f)$$

$$\mathbb{P}(Y \in \mathrm{Disc}(g\circ f)) \leq \mathbb{P}(Y \in \mathrm{Disc}(f)) = \mu_Y(\mathrm{Disc}(f)) = \mu_X(\mathrm{Disc}\, f)$$
$$= \mathbb{P}(X \in \mathrm{Disc}\, f) = 0.$$

$(g\circ f)(Y_n) \to (g \circ f)(Y)$ a.s.

& both sides $\leq \sup|g|$

$\therefore$ DCT $\Rightarrow$ $\mathbb{E}[(g\circ f)(Y_n)] \to \mathbb{E}[(g\circ f)(Y)]$

$$\mathbb{E}[f(X_n)] = \int g \, d\mu_{f(X_n)} \qquad \int g \, d\mu_{f(X)} = \mathbb{E}[f(X)]$$

$\leftarrow g =$ 

$M = \sup|f|$. ///