Some sequences of probability measures have no weakly convergent subsequences.

Eg. $\mu_n = \delta_n$.

The one and only obstruction is tightness.

Theorem: (Prokhorov's Compactness Thm)
Let S be a separable metric space. If $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(S, \mathcal{B}(S))$, $\exists$ vaguely convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$.

Corollary: If $\{\mu_n\}_{n=1}^{\infty}$ is also tight, then $\exists$ weakly convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ whose limit μ is a probability measure.

Pf. Enumerate the rationals: $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$

Let $F_n = F_{\mu_n}$.

- $\{F_n(q_1)\}_{n=1}^\infty$ is a sequence in $[0,1]$.

  $\therefore \exists$ convergent subsequence $\{F_{m_1(k)}(q_1)\}_{k=1}^\infty$

- $\{F_{m_1(k)}(q_2)\}_{k=1}^\infty$ is a sequence in $[0,1]$.

  $\therefore \exists$ convergent subsequence $\{F_{m_2(k)}(q_2)\}_{k=1}^\infty$

$\vdots$

Construct $\{m_j(k)\}_{j,k=1}^\infty$ s.t. $m_j(\cdot)$ is a subseq of $m_{j-1}(\cdot)$, and

$$F_{m_j(k)}(q_j) \to G(q_j) \in [0,1] \quad \forall j \in \mathbb{N}.$$

Then $F_{m_k(k)} \to G$ on $\mathbb{Q}$.

we'd like this to be the CDF of a measure.
Needs to be $\uparrow$, right-continuous.

$F : \mathbb{R} \to \mathbb{R}$, $F(x) = \inf\{G(q) : q \in \mathbb{Q},\ q > x\}$

↳ Non-decreasing: If $x < y$, $q > y$ $\Rightarrow$

↳ Right-continuous: If $x_n \downarrow$ $\qquad F(x_n) \downarrow$

$$\therefore \lim_{n \to \infty} F(x_n) = \inf_n F(x_n)$$
$$= \inf_n \inf\{G(q) : q \in \mathbb{Q},\ q > x_n\}$$

Thus, $F - \lim_{x \to \infty} F(x)$ is the CDF of a measure μ on $\mathbb{R}$.

To prove $\mu_{n_k} \to_v \mu$, it suffices to show $F_{n_k}(b) - F_{n_k}(a) \to F(b) - F(a)$ $\forall a, b \in \mathrm{Cont}(F)$.

In fact, we'll show the stronger claim that

$$F_{n_k}(x) \to F(x) \quad \forall x \in \mathrm{Cont}(F).$$

Let $x \in \mathrm{Cont}(F)$. Let $q_j \uparrow x$, $r_j \downarrow x$, $q_j, r_j \in \mathbb{Q}$.