Some sequences of probability measures have no weakly convergent subsequences.

Eg. $\mu_n = \delta_n$. $\forall K \subset \mathbb{R}$ compact, $\delta_n(K) = 0$ $\forall$ large n.

$\therefore$ $\exists$ no tight subsequence.

The one and only obstruction is tightness.

Theorem: (Prokhorov's Compactness Thm)
Let S be a separable metric space. If $\{\mu_n\}_{n=1}^{\infty} \subset \mathrm{Prob}(S, \mathcal{B}(S))$, $\exists$ vaguely convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$.

We'll prove it when $S = \mathbb{R}$ — "Helly's Selection Thm"

Corollary: If $\{\mu_n\}_{n=1}^{\infty}$ is also tight, then $\exists$ weakly convergent subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ whose limit μ is a probability measure.

Pf. Enumerate the rationals: $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$

Let $F_n = F_{\mu_n}$.

- $\{F_n(q_1)\}_{n=1}^\infty$ is a sequence in $[0,1]$.

  $\therefore \exists$ convergent subsequence $\{F_{m_1(k)}(q_1)\}_{k=1}^\infty$

- $\{F_{m_1(k)}(q_2)\}_{k=1}^\infty$ is a sequence in $[0,1]$.

  $m_2 \subseteq m_1$ $\quad$ $\therefore \exists$ convergent subsequence $\{F_{m_2(k)}(q_2)\}_{k=1}^\infty$ & q_1

$\vdots$

Construct $\{m_j(k)\}_{j,k=1}^\infty$ s.t. $m_j(\cdot)$ is a subseq of $m_{j-1}(\cdot)$,

and
$$F_{m_j(k)}(q_j) \to G(q_j) \in [0,1] \quad \forall j \in \mathbb{N}.$$

Then $\underbrace{F_{m_k(k)}}_{n_k} \to G$ on $\mathbb{Q}$. $\quad$ b/c $\{m_k(k)\}_{k=j}^\infty$ is a subseq of $\{m_j(k)\}_{k=1}^\infty$

we'd like this to be the CDF of a measure

Needs to be $\uparrow$, right-continuous. $\quad F(x) := \inf\{G(q) : q \in \mathbb{Q}, q > x\}$

$F : \mathbb{R} \to \mathbb{R}$, $F(x) = \inf\{G(q) : q \in \mathbb{Q},\ q > x\}$

↳ Non-decreasing: If $x < y$, $q > y > x \Rightarrow q > x$

$$\therefore \{G(q) : q \in \mathbb{Q},\ q > y\} \subseteq \{G(q) : q \in \mathbb{Q},\ q > x\}$$

$$F(y) = \inf \{G(q) : q \in \mathbb{Q},\ q > y\} \geq \underbrace{\inf\{G(q) : q \in \mathbb{Q},\ q > x\}}_{} = F(x)$$

↳ Right-continuous: If $x_n \downarrow x$ $\quad F(x_n) \downarrow$

$$\therefore \lim_{n\to\infty} F(x_n) = \inf_n F(x_n)$$
$$= \inf_n \inf\{G(q) : q \in \mathbb{Q},\ q > x_n\}$$
$$= \inf\{G(q) : q \in \mathbb{Q},\ \exists n\ q > x_n\}$$
$$= \inf\{G(q) : q \in \mathbb{Q},\ q > x\}$$
$$= F(x)$$

Thus, $F - \lim_{x\to\infty} F(x)$ is the CDF of a measure μ on $\mathbb{R}$.

To prove $\mu_{n_k} \to_v \mu$, it suffices to show $F_{n_k}(b) - F_{n_k}(a) \to F(b) - F(a)$ $\forall a, b \in Cont(F)$.

In fact, we'll show the stronger claim that

$$F_{n_k}(x) \to F(x) \quad \forall x \in \mathrm{Cont}(F).$$

Let $x \in \mathrm{Cont}(F)$. Let $q_j \uparrow x$, $r_j \downarrow x$, $q_j, r_j \in \mathbb{Q}$.

$$q_j < x < r_j$$

$$\therefore \forall k, j \quad F_{n_k}(q_j) \le F_{n_k}(x) \le F_{n_k}(r_j)$$

$$F(q_j) = G(q_j) = \lim_{k\to\infty} F_{n_k}(q_j) \le \liminf_{k\to\infty} F_{n_k}(x) \le \limsup_{k\to\infty} F_{n_k}(x) \le \lim_{k\to\infty} F_{n_k}(r_j) = G(r_j) = F(r_j).$$

$$F(x) = \inf\{G(q) : q \in \mathbb{Q}, q > x\}$$

$$\left.\begin{array}{l} q_j \uparrow x \\ r_j \downarrow x \end{array}\right\} \in \mathrm{Cont}(F)$$

$$\therefore F(x) \le \liminf_{k\to\infty} F_{n_k}(x) \le \limsup_{k\to\infty} F_{n_k}(x) \le F(x)$$

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