We've seen that if $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$,

1. $\int g \, d\mu_n \to \int g \, d\mu \quad \forall g \in C_b(\mathbb{R}^d)$

iff 2. $\int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in C_c(\mathbb{R}^d)$

E.g. $\mu_n = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Does μ_n have a limit in some sense?

↳ If $f \in C_c(\mathbb{R})$,

Note: $\mu_n \not\to_w$

In fact, $\{\mu_n\}_{n \geq 1}^{\infty}$ possesses no weakly convergent subsequence.

What's going on?

Def: Let μ_n, μ be Borel measures on $\mathbb{R}^d$.
Say μ_n converges vaguely to μ, $\mu_n \to_v \mu$,
if $\int_{\mathbb{R}^d} f \, d\mu_n \to \int_{\mathbb{R}^d} f \, d\mu \quad \forall f \in C_c(\mathbb{R}^d)$.

It is possible to lose mass, but not gain it, under vague convergence. Indeed, if

$$1 < \mu(\mathbb{R}^d)$$

Let $f \in C_c(\mathbb{R}^d)$, $1 \geq f \geq \mathbb{1}_{B_R}$.

Def: Let S be a topological space.
A family $\Lambda \subseteq \mathrm{Prob}(S, \mathcal{B}(S))$ is called **tight**
if $\forall \varepsilon > 0$, $\exists K_\varepsilon \subseteq S$ compact s.t. $\mu(K_\varepsilon) \geq 1-\varepsilon$ $\forall \mu \in \Lambda$.

i.e. $\inf_{\mu \in \Lambda} \mu(K_\varepsilon) \geq 1-\varepsilon$.

Eg. We showed that if $\mu_n \to_w \mu$ on $\mathbb{R}^d$, $\lim_{R\uparrow\infty} \inf_n \mu_n(\bar{B}_R) = 1$

So we can take $K_\varepsilon = \bar{B}_R$ for some sufficiently large R.

I.e. weakly convergent sequences of probability measures on $\mathbb{R}^d$ are tight.

Eg. $\mu_n = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n$. $\{\mu_n\}_{n=1}^\infty$ is not tight.

Theorem: If $\mu_n \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $\mu_n \to_v \mu$ for some Borel measure, then $\mu(\mathbb{R}^d) = 1$ iff $\{\mu_n\}_{n=1}^\infty$ is tight (in which case $\mu_n \to_w \mu$).

Pf. $(\Rightarrow)$

$(\Leftarrow)$ Fix $\varepsilon > 0$, let K_ε be s.t. $\mu_n(K_\varepsilon) \geq 1-\varepsilon$ $\forall n$.

Let $f \in C_c(\mathbb{R}^d)$ with $\mathbb{1}_{K_\varepsilon} \leq f \leq 1$. Then

$$\mu(\mathbb{R}^d) = \int_{\mathbb{R}^d} 1 \, d\mu$$

Prop: Let $\mu_n \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let μ be a finite Borel measure on $\mathbb{R}$.
Let $F_n(t) = \mu_n(-\infty, t]$, $F(t) = \mu(-\infty, t]$.
Then $\mu_n \to_v \mu$ iff

$$F_n(b) - F_n(a) \to F(b) - F(a) \quad \forall a, b \in \mathrm{Cont}(F).$$

Pf. [HW].

Eg. $\mu_n = \frac{1}{2}(\delta_n + \delta_{-n})$

$\int f \, d\mu_n =$