We've seen that if $\mu_n, \mu \in Prob(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$,

1. $\int g \, d\mu_n \to \int g \, d\mu \quad \forall g \in C_b(\mathbb{R}^d)$ $\qquad \mu_n \to_w \mu$

iff 2. $\int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in C_c^\infty(\mathbb{R}^d)$

E.g. $\mu_n = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n \in Prob(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Does μ_n have a limit in some sense?

↳ If $f \in C_c(\mathbb{R})$, $f(x) = 0 \quad \forall |x| \geq M$ $\quad \int f \, d\mu_n = \int f \, \frac{1}{2} d\delta_0 = \frac{1}{2} f(0)$, $n \geq M$

$\mu_n \to \frac{1}{2}\delta_0$

Note: $\mu_n \not\to_w \frac{1}{2}\delta_0$ $\qquad 1 = \int 1 \, d\mu_n \not\to \int 1 \, d(\frac{1}{2}\delta_0) = \frac{1}{2}$.

In fact, $\{\mu_n\}_{n \geq 1}^\infty$ possesses no weakly convergent subsequence.

What's going on?

Def: Let μ_n, μ be Borel measures on $\mathbb{R}^d$.
Say μ_n converges vaguely to μ, $\mu_n \to_v \mu$,
if $\int_{\mathbb{R}^d} f \, d\mu_n \to \int_{\mathbb{R}^d} f \, d\mu \quad \forall f \in C_c(\mathbb{R}^d)$.

It is possible to lose mass, but not gain it, under vague convergence. Indeed, if

$$1 < \mu(\mathbb{R}^d) = \lim_{R \uparrow \infty} \mu(B_R) \qquad \exists R < \infty \text{ s.t. } \mu(B_R) > 1.$$

Let $f \in C_c(\mathbb{R}^d)$, $1 \geq f \geq \mathbb{1}_{B_R}$.

$$\int f \, d\mu_n \to \int f \, d\mu \geq \int \mathbb{1}_{B_R} \, d\mu = \mu(B_R) > 1.$$

$$\therefore \forall \text{ large } n, \quad \int f \, d\mu_n > 1$$

$$\mu_n(\mathbb{R}^d) = \int d\mu_n \quad (\geq \int f\, d\mu_n)$$

Def: Let S be a topological space.
A family $\Lambda \subseteq Prob(S, \mathcal{B}(S))$ is called **tight** if
$\forall \varepsilon > 0, \exists K_\varepsilon \subseteq S$ compact s.t. $\mu(K_\varepsilon) \geq 1-\varepsilon \quad \forall \mu \in \Lambda$.

i.e. $\inf_{\mu \in \Lambda} \mu(K_\varepsilon) \geq 1-\varepsilon$.

Eg. We showed that if $\mu_n \to_w \mu$ on $\mathbb{R}^d$, $\lim_{R \uparrow \infty} \inf_n \mu_n(\bar{B}_R) = 1$
So we can take $K_\varepsilon = \bar{B}_R$ for some sufficiently large R.
I.e. weakly convergent sequences of probability measures on $\mathbb{R}^d$ are tight.

Eg. $\mu_n = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n$. $\{\mu_n\}_{n=1}^\infty$ is <u>not</u> tight.
If K is compact, $\mu_n(K) \leq \frac{1}{2}$ $\forall$ large n.

Theorem: If $\mu_n \in Prob(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $\mu_n \to_v \mu$ for some Borel measure, then $\mu(\mathbb{R}^d) = 1$ iff $\{\mu_n\}_{n=1}^\infty$ is tight (in which case $\mu_n \to_w \mu$).

Pf. ($\Rightarrow$) $\mu(\mathbb{R}^d)=1 \Rightarrow \{\mu_n\}$ is tight (b/c $\mu_n \to_v \mu$) ✓

($\Leftarrow$) Fix $\varepsilon>0$, let K_ε be s.t. $\mu_n(K_\varepsilon) \geq 1-\varepsilon \ \forall n$.
Let $f \in C_c(\mathbb{R}^d)$ with $\mathbb{1}_{K_\varepsilon} \leq f \leq 1$. Then

$$\mu(\mathbb{R}^d) = \int_{\mathbb{R}^d} 1 \, d\mu \geq \int_{\mathbb{R}^d} f \, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} f \, d\mu_n$$

$$\geq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \mathbb{1}_{K_\varepsilon} \, d\mu_n$$

$$= \liminf_{n\to\infty} \mu_n(K_\varepsilon) \geq 1-\varepsilon.$$

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Prop: Let $\mu_n \in Prob(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let μ be a finite Borel measure on $\mathbb{R}$.
Let $F_n(t) = \mu_n(-\infty, t]$, $F(t) = \mu(-\infty, t]$.
Then $\mu_n \to_v \mu$ iff

$$F_n(b) - F_n(a) \to F(b) - F(a) \quad \forall a, b \in Cont(F).$$

Pf. [HW].

Eg. $\mu_n = \frac{1}{2}(\delta_n + \delta_{-n})$

$$\int f \, d\mu_n = \frac{1}{2}(f(n) + f(-n)) \to 0 \text{ as } n \to \infty \quad \forall f \in C_c(\mathbb{R})$$

$$0 = \int f \, d0 \leftarrow \text{zero measure.}$$

$\mu_n \to_v 0$. $F_0(t) \equiv 0$.

$$F_n(t) = \begin{cases} 0 & t < -n \\ \frac{1}{2} & -n \leq t < n \\ 1 & t \geq n \end{cases} \to \frac{1}{2}.$$