Last time, we saw:

The Portmanteau Theorem

Let S be a complete, separable metric space.
Let $\mu_n, \mu \in \mathrm{Prob}(S, \mathcal{B}(S))$. TFAE:

1. $\int f\,d\mu_n \to \int f\,d\mu \ \forall f \in C_b(S)$. I.e. $\mu_n \to_w \mu$.
2. $\int f\,d\mu_n \to \int f\,d\mu \ \forall f \in \mathrm{Lip}_b(S)$
3. $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ $\forall$ closed $F \subseteq S$.
4. $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ $\forall$ open $G \subseteq S$.
5. $\mu_n(A) \to \mu(A)$ $\forall$ μ-continuity sets $A \in \mathcal{B}(S)$.

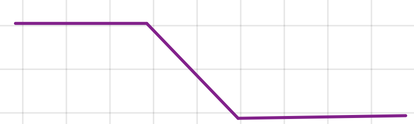
In this lecture, we'll add 3 more equivalent conditions in the most relevant cases that $S = \mathbb{R}^d$.

Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $\mu_n \to_w \mu$

iff $\int_{\mathbb{R}^d} f \, d\mu_n \to \int_{\mathbb{R}^d} f \, d\mu \quad \forall f \in C_c(\mathbb{R}^d)$.

Lemma: If   holds true, then

$$\lim_{R \uparrow \infty} \inf_n \mu_n(\bar{B}_R) = 1$$

Pf. Let $g_R =$ 

$\varphi_R(x) = g_R(|x|)$ $\therefore$ $\mathbb{1}_{\bar{B}_{R/2}} \leq \varphi_R \leq \mathbb{1}_{\bar{B}_R}$

$\therefore$ $\int \varphi_R \, d\mu_n \leq \int \mathbb{1}_{\bar{B}_R} \, d\mu_n = \mu_n(\bar{B}_R)$

$\mu(\bar{B}_{R/2}) \uparrow 1$ as $R \uparrow \infty$. $\therefore$ $\forall \alpha \in (0,1)$, $\mu(\bar{B}_R) > \alpha$ for all large R.

Theorem: Let $\mu_n, \mu \in \text{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $\mu_n \to_w \mu$ iff $\int_{\mathbb{R}^d} f \, d\mu_n \to \int_{\mathbb{R}^d} f \, d\mu \quad \forall f \in C_c(\mathbb{R}^d)$.

Pf. ($\Rightarrow$) $C_c \subseteq C_b$

($\Leftarrow$) Let $f \in C_b(\mathbb{R}^d)$, and set $f_R = f \cdot \varphi_R$

$$\limsup_{n\to\infty} \left| \int f \, d\mu - \int f \, d\mu_n \right|$$

$$\leq \limsup_{n\to\infty} \left| \int f \, d\mu - \int f_R \, d\mu \right| + \limsup_{n\to\infty} \left| \int f_R \, d\mu - \int f_R \, d\mu_n \right| + \limsup_{n\to\infty} \left| \int f_R \, d\mu_n - \int f \, d\mu_n \right|$$

Cor: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $\mu_n \to_w \mu$ iff $\int_{\mathbb{R}^d} f \, d\mu_n \to \int_{\mathbb{R}^d} f \, d\mu \quad \forall f \in C_c^\infty(\mathbb{R}^d)$.

Pf. Fix a smooth probability density ρ on $\mathbb{R}^d$ s.t. $0 \le \rho \le \mathbb{1}_{\bar{B}_1}$.
For $\varepsilon > 0$, $f \in C_c(\mathbb{R}^d)$, set

$$f_\varepsilon(x) = \int_{\mathbb{R}^d} f(x + \varepsilon u)\, \rho(u)\, du$$

Note: $\sup_{x \in \mathbb{R}^d} |f(x) - f_\varepsilon(x)| = \sup_x \left| \int f(x) \rho(u) du - \int f(x + \varepsilon u) \rho(u) du \right|$

$\therefore \limsup_{n \to \infty} \left| \int f d\mu - \int f d\mu_n \right| \leq \limsup_{n \to \infty} \left[\left| \int f d\mu - \int f_\varepsilon d\mu \right| + \left| \int f_\varepsilon d\mu - \int f_\varepsilon d\mu_n \right| + \left| \int f_\varepsilon d\mu_n - \int f d\mu_n \right| \right]$

For any function $F: S \to T$ between topological spaces,

$$\mathrm{Cont}(F) := \{x \in S : F \text{ is continuous @ } F\}$$

$$\mathrm{Disc}(F) := S \setminus \mathrm{Cont}(F).$$

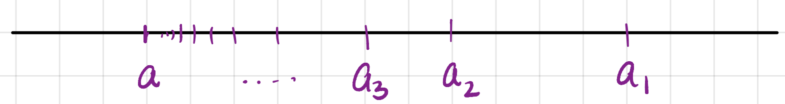
Recall that every probability measure on $\mathbb{R}$ is a Stieltjes measure:

$$\mu = \mu_F \quad \text{where } F(t) = \mu(-\infty, t] \text{ is right-continuous, } \uparrow.$$

Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R})$. Set $F_n(t) = \mu_n(-\infty, t]$, $F(t) = \mu(-\infty, t]$.

Then $\mu_n \rightharpoonup \mu$ iff $F_n(t) \to F(t) \ \forall t$

Eg. $a_n \downarrow a$ (strict) in $\mathbb{R}$.



Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R})$. Set $F_n(t) = \mu_n(-\infty, t]$, $F(t) = \mu(-\infty, t]$.

Then $\mu_n \to_w \mu$ iff $F_n(t) \to F(t)$ $\forall t \in \mathrm{Cont}(F)$.

Pf. ($\Rightarrow$) If $t \in \mathrm{Cont}(F)$, $\mu(\{t\}) = 0$.

($\Leftarrow$) $F \uparrow$, so $\mathrm{Disc}(F)$ is countable. Thus $\mathrm{Cont}(F)$ is dense in $\mathbb{R}$.

Let $f \in C_c(\mathbb{R})$. Then f is uniformly continuous.

Fix $\varepsilon > 0$, and let $\delta > 0$ be s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$.

Let $\pi = \{x_0 < x_1 < \cdots < x_k\}$ be a partition of $\mathrm{supp} f$ s.t. $x_j \in \mathrm{Cont}(F)$ $\forall j$,

and $|x_j - x_{j-1}| < \delta$ $\forall j$.

$$f_\pi := \sum_{j=1}^{k} f(x_j) \mathbb{1}_{(x_{j-1}, x_j]}$$

$$\therefore \quad \limsup_{n\to\infty} \left| \int f \, d\mu_n - \int f \, d\mu \right|$$

$$\leq \limsup_{n\to\infty} \left[\left| \int f \, d\mu - \int f_\pi \, d\mu \right| + \left| \int f_\pi \, d\mu - \int f_\pi \, d\mu_n \right| + \left| \int f_\pi \, d\mu_n - \int f \, d\mu_n \right| \right]$$