Last time, we saw:

The Portmanteau Theorem

Let S be a complete, separable metric space.
Let $\mu_n, \mu \in Prob(S, \mathcal{B}(S))$. TFAE:

1. $\int f\,d\mu_n \to \int f\,d\mu \;\; \forall f \in C_b(S)$. I.e. $\mu_n \to_w \mu$.
2. $\int f\,d\mu_n \to \int f\,d\mu \;\; \forall f \in Lip_b(S)$
3. $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F) \;\; \forall$ closed $F \subseteq S$.
4. $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G) \;\; \forall$ open $G \subseteq S$.
5. $\mu_n(A) \to \mu(A) \;\; \forall$ μ-continuity sets $A \in \mathcal{B}(S)$.

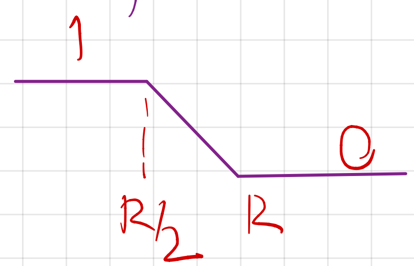
In this lecture, we'll add 3 more equivalent conditions in the most relevant cases that $S = \mathbb{R}^d$.

Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $\mu_n \to_w \mu$

$$\text{iff } \underbrace{\int_{\mathbb{R}^d} f\, d\mu_n \to \int_{\mathbb{R}^d} f\, d\mu \quad \forall f \in C_c(\mathbb{R}^d)}. \subseteq C_b(\mathbb{R}^d)$$

Lemma: If ___ holds true, then

$$\lim_{R\uparrow\infty} \inf_n \mu_n(\bar{B}_R) = 1$$

Pf. Let $g_R =$ 

$$g_R(t) = \begin{cases} 1, & t \le R/2 \\ 1 - \frac{t - R/2}{R/2}, & R/2 \le t \le R \\ 0, & t \ge R \end{cases}$$

$\psi_R(x) = g_R(|x|)$ $\therefore$ $\mathbb{1}_{\bar{B}_{R/2}} \le \psi_R \le \mathbb{1}_{\bar{B}_R}$

$\therefore$ $\int \psi_R\, d\mu_n \le \int \mathbb{1}_{\bar{B}_R}\, d\mu_n = \mu_n(\bar{B}_R)$

$$\liminf_{n\to\infty} \mu_n(\bar{B}_R) \ge \liminf_{n\to\infty} \int \psi_R\, d\mu_n = \int \psi_R\, d\mu \ge \mu(\bar{B}_{R/2})$$

$\mu(\bar{B}_{R/2}) \uparrow 1$ as $R \uparrow \infty$. $\therefore$ $\forall \alpha \in (0,1)$, $\mu(\bar{B}_R) > \alpha$ for all large R.

$\therefore$ $\exists N_\alpha$ s.t. $\forall n \ge N_\alpha$, $\mu_n(\bar{B}_R) \ge \alpha$. Also $\mu_k(\bar{B}_R) \uparrow 1$ $\forall k \le N_\alpha$ increasing past R_k $1 \le k \le N_\alpha$, $\liminf_{n\to\infty} \mu_n(\bar{B}_R) \ge \alpha$.

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Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $\mu_n \to_w \mu$ iff $\int_{\mathbb{R}^d} f\, d\mu_n \to \int_{\mathbb{R}^d} f\, d\mu \quad \forall f \in C_c(\mathbb{R}^d)$.

Pf. $(\Rightarrow)$ $C_c \subseteq C_b$

$(\Leftarrow)$ Let $f \in C_b(\mathbb{R}^d)$, and set $f_R = f \cdot \varphi_R \in C_c(\mathbb{R}^d)$

$f_R \uparrow f$ as $R \uparrow \infty$

$$\limsup_{n\to\infty} \left| \int f\,d\mu - \int f\,d\mu_n \right|$$

$$\leq \cancel{\limsup_{n\to\infty}} \left| \int f\,d\mu - \int f_R\,d\mu \right| + \underbrace{\limsup_{n\to\infty} \left| \int f_R\,d\mu - \int f_R\,d\mu_n \right|}_{\to 0 \text{ as } n\to\infty} + \limsup_{n\to\infty} \left| \int f_R\,d\mu_n - \int f\,d\mu_n \right|$$

$\left| \int f\,d\mu - \int f_R\,d\mu \right| \to 0$ as $R \to \infty$.

$$\leq \limsup_{n\to\infty} \int \underbrace{|f - f_R|}_{|f|(1-\varphi_R) \searrow 0} d\mu_n \leq \sup|f| \cdot \sup_n \int (1-\varphi_R)\,d\mu_n$$

$1 - \varphi_R \leq \mathbb{1}_{\mathbb{R}^d \setminus \bar{B}_{R/2}}$

$$\leq \sup_n \mu_n(\mathbb{R}^d \setminus \bar{B}_{R/2}) = 1 - \inf_n \mu_n(\bar{B}_{R/2}) \to 0 \text{ as } R \to \infty.$$

Cor: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then $\mu_n \to_w \mu$

iff $\int_{\mathbb{R}^d} f \, d\mu_n \to \int_{\mathbb{R}^d} f \, d\mu \quad \forall f \in C_c^\infty(\mathbb{R}^d) \subseteq C_c(\mathbb{R}^d)$

Pf. Fix a smooth probability density ρ on $\mathbb{R}^d$ s.t. $0 \le \rho \le \mathbb{1}_{\bar{B}_1}$.

For $\varepsilon > 0$, $f \in C_c(\mathbb{R}^d)$, set

$y = x + \varepsilon u$

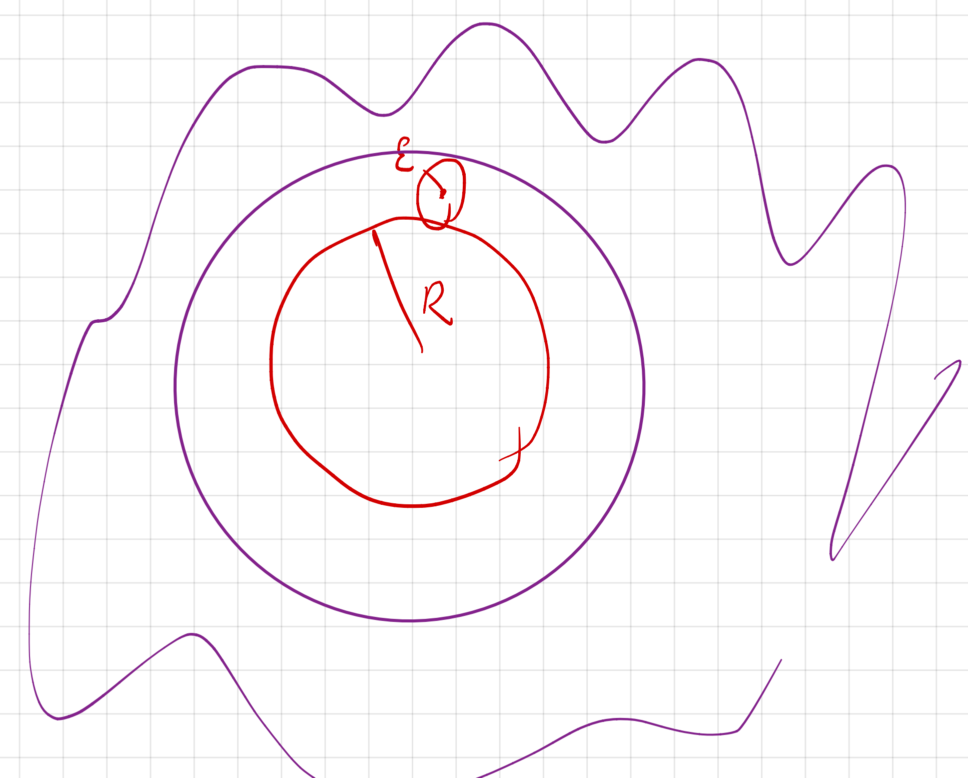
$$f_\varepsilon(x) = \int_{\mathbb{R}^d} f(x + \varepsilon u)\, \rho(u)\, du$$

$$= \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} f(y)\, \rho\left(\frac{y-x}{\varepsilon}\right) dy \quad \therefore \in C^\infty$$

$\leftarrow$ diff under $\int$ repeatedly

$v = \varepsilon u$

$$f_\varepsilon(x) = \frac{1}{\varepsilon^d} \int_{\substack{\cancel{\mathbb{R}^d} \\ \bar{B}_\varepsilon}} f(x+v)\, \rho(v/\varepsilon)\, dv$$



$\operatorname{supp} f_\varepsilon \subseteq \bar{B}_{R+\varepsilon} \quad \therefore f_\varepsilon \in C_c^\infty$.

Note: $\sup_{x\in\mathbb{R}^d} |f(x) - f_\varepsilon(x)| = \sup_x \left| \int f(x)\rho(u)\,du - \int f(x+\varepsilon u)\rho(u)\,du \right|$

M_ε

$$\leq \sup_x \int |f(x) - f(x+\varepsilon u)|\,\rho(u)\,du$$

$$\leq \sup_x \sup_{u\in B_1} |f(x) - f(x+\varepsilon u)| \underbrace{\int \rho(u)\,du}_{1} \longrightarrow 0.$$

$$\therefore \limsup_{n\to\infty} \left| \int f\,d\mu - \int f\,d\mu_n \right| \leq \limsup_{n\to\infty} \left[\underbrace{\left| \int f\,d\mu - \int f_\varepsilon\,d\mu \right|}_{\leq M_\varepsilon} + \underbrace{\left| \int f_\varepsilon\,d\mu - \int f_\varepsilon\,d\mu_n \right|}_{\to 0 \text{ as } n\to\infty \text{ by assump.}} + \underbrace{\left| \int f_\varepsilon\,d\mu_n - \int f\,d\mu_n \right|}_{\leq M_\varepsilon} \right]$$

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For any function $F : S \to T$ between topological spaces,

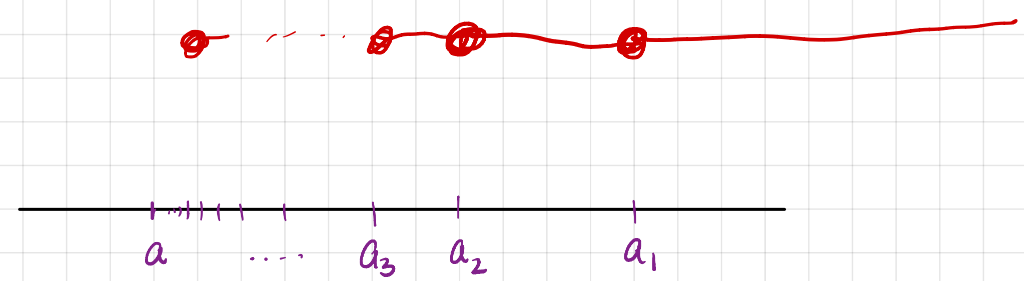
$$\mathrm{Cont}(F) := \{x \in S : F \text{ is continuous @ } F\}$$
$$\mathrm{Disc}(F) := S \setminus \mathrm{Cont}(F).$$

Recall that every probability measure on $\mathbb{R}$ is a Stieltjes measure:

$$\mu = \mu_F \quad \text{where } F(t) = \mu(-\infty, t] \text{ is right-continuous, } \uparrow.$$

Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R})$. Set $F_n(t) = \mu_n(-\infty, t]$, $F(t) = \mu(-\infty, t]$.

Then $\mu_n \to_w \mu$ iff $F_n(t) \to F(t)$ $\forall t \in \mathrm{Cont}(F)$



Eg. $a_n \downarrow a$ (strict) in $\mathbb{R}$.

$$F_n(t) = \delta_{a_n}(-\infty, t] = \mathbb{1}_{t \geq a_n}$$
$$F(t) = \delta_a(-\infty, t] = \mathbb{1}_{t \geq a}$$

$$\lim_{n\to\infty} F_n(t) = 0 \quad \text{iff} \quad t < a$$
$$\lim_{n\to\infty} F_n(t) = 1 \quad \text{iff} \quad t > a$$

But $\lim_{n\to\infty} F_n(a) = 0$.

Theorem: Let $\mu_n, \mu \in \mathrm{Prob}(\mathbb{R})$. Set $F_n(t) = \mu_n(-\infty, t]$, $F(t) = \mu(-\infty, t]$.
Then $\mu_n \to_w \mu$ iff $F_n(t) \to F(t)$ $\forall t \in \mathrm{Cont}(F)$.

Pf. ($\Rightarrow$) If $t \in \mathrm{Cont}(F)$, $\mu(\{t\}) = 0$. $\{t\} = \partial(-\infty, t]$ $\therefore$ $(-\infty, t]$ is μ-continuous.
$\therefore$ by Portmanteau (5), $F_n(t) = \mu_n(-\infty, t] \to \mu(-\infty, t] = F(t)$ ✓

($\Leftarrow$) $F \uparrow$, so $\mathrm{Disc}(F)$ is countable. Thus $\mathrm{Cont}(F)$ is dense in $\mathbb{R}$.
Let $f \in C_c(\mathbb{R})$. Then f is uniformly continuous.
Fix $\varepsilon > 0$, and let $\delta > 0$ be s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$.
Let $\pi = \{x_0 < x_1 < \cdots < x_k\}$ be a partition of $\mathrm{supp} f$ s.t. $x_j \in \mathrm{Cont}(F)$ $\forall j$,
and $|x_j - x_{j-1}| < \delta$ $\forall j$.

$$f_\pi := \sum_{j=1}^{k} f(x_j) \mathbb{1}_{(x_{j-1}, x_j]}$$

$\therefore \sup_{x \in \mathbb{R}} |f(x) - f_\pi(x)| < \varepsilon/2$

$\therefore \int f_\pi \, d\mu_n \to \int f_\pi \, d\mu$.

$$\therefore \left| \int f \, d\mu_n - \int f_\pi \, d\mu_n \right| \le \int |f - f_\pi| \, d\mu_n \le \varepsilon/2$$

$$\int f_\pi \, d\mu_n = \sum_{j=1}^{k} f(x_j) \mu_n(x_{j-1}, x_j] \leftarrow F_n(x_j) - F_n(x_{j-1}) \to F(x_j) - F(x_{j-1}) = \mu(x_{j-1}, x_j]$$

$$\therefore \limsup_{n\to\infty} \left| \int f\, d\mu_n - \int f\, d\mu \right| \le \varepsilon \quad \forall \varepsilon > 0.$$

$$\le \limsup_{n\to\infty} \Big[\underbrace{\left| \int f\, d\mu - \int f_\pi\, d\mu \right|}_{\le \varepsilon/2} + \underbrace{\left| \int f_\pi\, d\mu - \int f_\pi\, d\mu_n \right|}_{\to 0 \text{ as } n\to\infty} + \underbrace{\left| \int f_\pi\, d\mu_n - \int f\, d\mu_n \right|}_{\le \varepsilon/2} \Big]$$

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