In non-discrete settings, total variation convergence is usually too much to ask.

Eg. Let $a_n \in \mathbb{R}$, $a_n \to a$. Surely $\delta_{a_n} \to \delta_a$?

$$d_{TV}(\delta_{a_n}, \delta_a) = \sup_B |\delta_{a_n}(B) - \delta_a(B)| \geq |\underbrace{\delta_{a_n}(\{a\})}_{0} - \underbrace{\delta_a(\{a\})}_{=1}| = 1 \text{ i.o.}$$

In general, if $\mu \perp \nu$, $d_{TV}(\mu, \nu) \geq |\mu(A) - \nu(A)| = 1$,

$\exists A \;\; \mu(A) = 0, \; \nu(A) = 1.$

Eg. A discrete approximation of $\mathrm{Unif}([0,1])$:

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{k/n} \qquad d_{TV}(\mu_n, \mathrm{Unif}[0,1]) = 1.$$

Why do we think $\frac{1}{n}\sum_{k=1}^{n}\delta_{k/n} \to \mathrm{Unif}([0,1])$ should be true?

Recall from last lecture: $\left|\int h\,d\mu - \int h\,d\nu\right| \leq 2\,d_{TV}(\mu,\nu)\cdot\sup|h|$

$\therefore\ d_{TV}(\mu_n,\mu) \to 0 \Rightarrow \int h\,d\mu_n \to \int h\,d\mu$

If $\mu_n = \frac{1}{n}\sum_{k=1}^{n}\delta_{k/n}$, $\int h\,d\mu_n = \sum_{k=1}^{n} h\left(\frac{k}{n}\right)\cdot\frac{1}{n}$ $\leftarrow$ Riemann sum for $\int_0^1 h(x)\,\lambda(dx)$

$\therefore$ If h is Riemann int'ble

$h = \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ $\qquad h\left(\frac{k}{n}\right) = 1.$

$\int h\,d\mu_n = 1. \qquad \int h\,d\lambda = 0.$

Def: Let S be a metric space, $\mu_n, \mu \in \mathrm{Prob}(S, \mathcal{B}(S))$.
Say μ_n converges weakly to μ, $\mu_n \xrightarrow{w} \mu$ or $\mu_n \Rightarrow \mu$,
if $\int f\,d\mu_n \to \int f\,d\mu \quad \forall f \in C_b(S).$

Eg. If $a_n \to a$, then $\int f \, d\delta_{a_n} = f(a_n) \to f(a) = \int f \, d\delta_a \quad \forall f \in C_b$

Prop: If $d_{TV}(\mu_n, \mu) \to 0$, then $\mu_n \to_w \mu$.

Pf. $\Downarrow$

$\int h \, d\mu_n \to \int h \, d\mu \quad \forall$ bdd mbl. h ✓

Restricting to continuous test functions $f \in C_b$ allows μ_n to "smear out".

Notation: If X_n, X are S-valued random variables, say $X_n \to_w X$ iff $\mu_{X_n} \to_w \mu_X$.

Prop: If $X_n, X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B}(S))$ and $X_n \to_{\mathbb{P}} X$, then $X_n \to_w X$.

First, a **Lemma**: If $X_n \to_{\mathbb{P}} X$ and $g \in C(S)$, then $g(X_n) \to_{\mathbb{P}} g(X)$.

Pf. For $\varepsilon, \delta > 0$, let $B_{\varepsilon,\delta}(g) = \{ x \in S : \exists y \in S \ d(x,y) < \delta, \ |g(x) - g(y)| \geq \varepsilon \}$

Continuity of g means that, for fixed $\varepsilon > 0$, $B_{\varepsilon,\delta}(g) \downarrow \emptyset$ as $\delta \downarrow 0$.

$$\{ |g(X_n) - g(X)| \geq \varepsilon \} \subseteq \{ d(X_n, X) \geq \delta \} \cup \{ X \in B_{\varepsilon,\delta}(g) \}$$

$$\mathbb{P}(|g(X_n) - g(X)| \geq \varepsilon) \leq \underbrace{\mathbb{P}(d(X_n, X) \geq \delta)}_{\downarrow n \to \infty \ 0} + \underbrace{\mathbb{P}(X \in B_{\varepsilon,\delta}(g))}_{= \mu_X(B_{\varepsilon,\delta}(g)) < \eta/2}$$

$\eta/2$

///

Prop: If $X_n, X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{B}(S))$ and $X_n \to_{\mathbb{P}} X$, then $X_n \to_w X$.

Pf. Let $f \in C_b(S)$. Then

$$\int f \, d\mu_{X_n} = \mathbb{E}[f(X_n)]$$

$X_n \to_{\mathbb{P}} X$, $\therefore$ $f(X_n) \to_{\mathbb{P}} f(X)$.

$|f(X_n)| \leq M \quad \forall n$.

$\therefore$ by DCT [Driver, Cor 17.10] $\therefore$ $f(X_n) \to f(X)$ in L^1.

$$\therefore \mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)] = \int f \, d\mu_X .$$

///

Cor: If $X_n \to X$ a.s., or if $X_n \to X$ in L^p, then $X_n \to_w X$.

What was fundamentally wrong in the example $d_{TV}(\delta_{a_n}, \delta_a) \not\to 0$?

d_{TV} is too sensitive to jumps - to "discontinuity sets".

Def: Let μ be a Borel measure on a metric space S.
An event $B \in \mathcal{B}(S)$ is a continuity set for μ if

$$\mu(\partial A) = 0. \qquad \partial A = \bar{A} \setminus \mathrm{int}(A)$$

$$\Updownarrow$$

$$\mu(A) = \mu(\bar{A}) = \mu(\mathrm{int}(A))$$

Eg. $(-\infty, a]$ is not a continuity set for δ_a

$$\delta_a(\partial(-\infty, a]) = \delta_a(\{a\}) = 1 \neq 0.$$

Eg. If $\mu \in \mathrm{Prob}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and F_μ is continuous (i.e. $\mu(\{a\}) = 0 \ \forall a \in \mathbb{R}$)
then all intervals are continuity sets for μ.

Eg. $\mathbb{Q} \cap [0,1]$ not cont. set for λ

$$\partial(\mathbb{Q} \cap [0,1]) = \overline{\mathbb{Q} \cap [0,1]} \setminus \mathrm{int}(\mathbb{Q} \cap [0,1]) = [0,1]$$

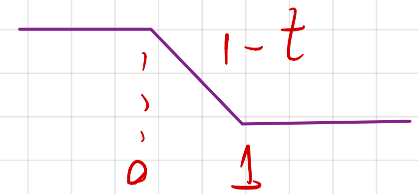
The Portmanteau Theorem

Let S be a complete, separable metric space.

Let $\mu_n, \mu \in \mathrm{Prob}(S, \mathcal{B}(S))$. TFAE:

1. $\int f \, d\mu_n \to \int f \, d\mu \ \forall f \in C_b(S)$. I.e. $\mu_n \to_w \mu$.
2. $\int f \, d\mu_n \to \int f \, d\mu \ \forall f \in \mathrm{Lip}_b(S)$ (bounded Lipschitz functions)
3. $\limsup_{n \to \infty} \mu_n(F) \leq \mu(F) \ \forall$ closed $F \subseteq S$.
4. $\liminf_{n \to \infty} \mu_n(G) \geq \mu(G) \ \forall$ open $G \subseteq S$.
5. $\mu_n(A) \to \mu(A) \ \forall$ μ-continuity sets $A \in \mathcal{B}(S)$.

Pf. $(1 \Rightarrow 2)$: $\mathrm{Lip}_b(S) \subseteq C_b(S)$ ✓

$(2 \Rightarrow 3)$: ✓ Let $\psi =$ 

$\psi \in \mathrm{Lip}_b(\mathbb{R})$ $\|\psi\|_{Lip} \leq 1$

Fix a closed set F. Set $f_k(x) = \psi(k \cdot \underbrace{d(x,F)}_{\inf_{y \in F} d(x,y)})$ $\leftarrow$ Lip-1 of x.

$$|f_k(x) - f_k(y)| \leq k\,|d(x,F) - d(y,F)| \leq k\,d(x,y)$$

$\therefore f_k \in \mathrm{Lip}(S)$ $\|f_k\|_{Lip} \leq k$. $|f_k| \leq 1$ $f_k \in \mathrm{Lip}_b$.

$\therefore$ by (2), $\int f_k \, d\mu_n \to \int f_k \, d\mu$.

As $k \to \infty$: if $d(x,F) > 0$, $f_k(x) \downarrow \psi(\infty) = 0$

if $d(x,F) = 0$, $f_k(x) = \psi(0) = 1$.

$\therefore f_k \downarrow \mathbb{1}_F = \mathbb{1}_F$

Thus $\limsup_{n \to \infty} \mu_n(F) = \limsup_{n \to \infty} \int \mathbb{1}_F \, d\mu_n \leq \limsup_{n \to \infty} \int f_k \, d\mu_n = \int f_k \, d\mu \overset{\substack{MCT \\ DCT}}{\downarrow} \int \mathbb{1}_F \, d\mu = \mu(F)$

$(3 \Rightarrow 4)$: ✓ $1 - \liminf_{n\to\infty} \mu_n(G) = \limsup_{n\to\infty} (1 - \mu_n(G))$

$= \limsup_{n\to\infty} \mu_n(G^c) \overset{(3)}{\leq} \mu(G^c) = 1 - \mu(G)$

$(3, 4 \Rightarrow 5)$ If $\mu(\partial A) = 0$, $\therefore \mu(A) = \mu(\bar{A}) = \mu(\mathrm{Int}(A))$,

$\therefore \limsup_{n\to\infty} \mu_n(A) \leq \limsup_{n\to\infty} \mu_n(\bar{A}) \overset{(3)}{\leq} \mu(\bar{A}) = \mu(A)$

& $\liminf_{n\to\infty} \mu_n(A) \geq \liminf_{n\to\infty} \mu_n(\mathrm{Int}(A)) \overset{(4)}{\geq} \mu(\mathrm{Int}(A)) = \mu(A)$

$\lim_{n\to\infty} \mu_n(A) = \mu(A)$.

It remains to prove (S ⇒ 1): If $\mu_n(A) \to \mu(A)$ $\forall \mu$-continuous A,

then $\int f d\mu_n \to \int f d\mu$ $\forall f \in C_b$.

Let $f \in C_b(S)$; so $a \le f(x) \le b$ $\forall x \in S$ for some $a \le b$ in $\mathbb{R}$.

For each $x \in S$, $f(x) - a = \lambda([a, f(x)]) = \int_a^{f(x)} dt = \int_a^b \mathbb{1}_{[a, f(x)]}(t) dt = \int_a^b \mathbb{1}_{f(x) \ge t} dt$

$$\therefore \int_S f d\mu_n = a + \int_S (f-a) d\mu_n = a + \int_S \left(\int_a^b \mathbb{1}_{f \ge t} dt \right) d\mu_n$$

$$\overset{\text{Tonelli}}{=} a + \int_a^b \Big(\underbrace{\int_S \mathbb{1}_{t \le f} d\mu_n}_{\mu_n\{f \ge t\}} \Big) dt$$

Similarly, $\int_S f d\mu = a + \int_a^b \mu\{f \ge t\} dt$

$\therefore$ Suffices to show that $\mu_n\{f \ge t\} \to \mu\{f \ge t\}$ for $[\lambda]$ a.e. $t \in \mathbb{R}$. //

By assumption (S), this holds true except on $E = \{t \in [a,b] : \mu(\partial\{f \ge t\}) > 0\}$.

$E \subseteq \{t \in [a,b] : \mu\{f = t\} > 0\}$ ⟵ countable

$\mu(S) = \mu(\biguplus_t \{f = t\})$

$\{f = t\}$

$\{f > t\}$ open