The Lebesgue Integral for $\overline{\mathbb{R}}$-Valued Functions

Let $f:(\Omega,\mathcal{F},\mu)\to(\overline{\mathbb{R}},\mathcal{B}(\overline{\mathbb{R}}))$ be measurable.

$$f_+ = f\,\mathbb{1}_{\{f\geq 0\}} \qquad f_- = -f\,\mathbb{1}_{\{f\leq 0\}} \qquad f = f_+ - f_-$$

Def: f is called integrable, $f\in\mathcal{L}^1(\Omega,\mathcal{F},\mu)$, if

$$(\star) \qquad \int f_\pm \, d\mu < \infty .$$

In this case, we define

$$\int f \, d\mu :=$$

Note: since $|f| = f_+ + f_-$, alternatively we have

$$\mathcal{L}^1(\Omega,\mathcal{F},\mu) =$$

Note also: b/c $(\star)$, $f_\pm < \infty$ a.s. Therefore, we can restrict our attention to the complement of a nullset and assume f is $\mathbb{R}$-valued.

Proposition: [10.20] $\mathcal{L}^1(\Omega,\mathcal{F},\mu)$ is a $\mathbb{R}$-vector space,

$\int : \mathcal{L}^1(\Omega,\mathcal{F},\mu) \to \mathbb{R}$ is linear, and

$f \le g$ a.s. $\Rightarrow \int f\,d\mu \le \int g\,d\mu \quad \forall f,g \in \mathcal{L}^1$.

Pf.

$f \leq g$ a.s. $\Rightarrow$ $\int f \, d\mu \leq \int g \, d\mu \quad \forall f, g \in \mathcal{L}^1$.

Bonus: $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$.

Now we know how to "compute" expectations:

if $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ then $\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$.

(As to how to actually compute it, we'll discuss that in the following lectures.)

L^1

Recall the pseudo-metric $d_\mu(A,B) = \mu^*(A \triangle B)$

if $A, B \in \mathcal{F}$: $= \mu(A \triangle B)$

We can think of $\mathcal{F}$ "$\subset$" $L^+(\mathcal{F})$ by $A \mapsto \mathbb{1}_A$.

This presents a natural way to extend d_μ to $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ "$\supset$" $\mu^{-1}([0,\infty))$

Def: The L^1-norm $\| \; \|_{L^1}$ is defined on $\mathcal{L}^1$ by

$$\|f\|_{L^1} := \int_\Omega |f| \, d\mu$$

A norm on a vector space V is a function $\|\cdot\|: V \to [0,\infty)$ s.t.

- $\|\alpha f\| = |\alpha| \, \|f\| \quad \forall f \in V, \ \alpha \in \mathbb{R}$
- $\|f+g\| \leq \|f\| + \|g\| \quad \forall f, g \in V$
- $\|f\| = 0$ if and only if $f = 0$.

if this part is missing, $\| \ \|$ is a seminorm.

If $\| \ \|$ is a norm, then $d(f,g) = \|f-g\|$ is a metric.

$\| \ \|_{L^1}$ is a seminorm on $\mathcal{L}^1$.

But it is not a norm.

[10.11]

$\|f\|_{L^1} = 0 \iff \int |f| \, d\mu = 0 \iff |f| = 0 \text{ a.s.} [\mu] \iff f = 0 \text{ a.s.} [\mu].$

Def: The relation $f \sim_\mu g$ iff $f = g$ a.s.$[\mu]$ is an equivalence relation on $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$.

$$L^1(\Omega, \mathcal{F}, \mu) := \mathcal{L}^1(\Omega, \mathcal{F}, \mu) / \sim_\mu$$

$\uparrow$

Elements are equivalence classes $[f]_{\sim_\mu}$ s.t. $f_1, f_2 \in [f]_{\sim_\mu} \Leftrightarrow f = g$ a.s.$[\mu]$.

Given $[f] \in L^1(\Omega, \mathcal{F}, \mu)$, a function $f_1 \in [f]$ is called a version of $[f]$.

Note: if $f_1 \sim_\mu f_2$, $\int f_1 \, d\mu = \int f_2 \, d\mu$. Thus, $\int [f] \, d\mu := \int f_1 \, d\mu$

[10.11] for any $f_1 \in [f]$ makes sense.

All the properties of $\int \cdot \, d\mu$ on $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ descend nicely to $L^1(\Omega, \mathcal{F}, \mu)$.

And, even better:

$\| \ \|_{L^1}$ is a genuine norm on L^1.

$\therefore$ $d_\mu([f], [g]) = \| [f] - [g] \|_{L^1} = \| [f-g] \|_{L^1} = \int |f-g| \, d\mu$ is a genuine metric on L^1.

Going forward: we forget $\mathcal{L}^1$,
and treat the elements of L^1 as functions.

(Just keep in mind the "versions" business.)

We can also define L^p-norms: for $1 \le p < \infty$,

$$L^p(\Omega, \mathcal{F}, \mu) = (\sim_\mu \text{ equiv. classes of}) \left\{ \text{meas. } f : \Omega \to \mathbb{R} \text{ s.t. } \int |f|^p < \infty \right\}$$

$$\|f\|_{L^p} := \left(\int |f|^p \, d\mu \right)^{1/p}.$$

Theorem: [17.27] $L^p(\Omega, \mathcal{F}, \mu)$ is a complete normed space.

We will come back to L^p and completeness (and extending to $p = \infty$) later when we talk about different modes of convergence of random variables.

For L^+, we have two integral convergence results: the MCTheorem, and Fatou's Lemma.
(Both can be partially extended to L^1, but they are not sufficiently powerful for most applications.)

The Dominated Convergence Theorem (DCT) [10.28]

Suppose $f_n, g_n, g \in L^1$, with

- $f_n \to f$ a.s. and $g_n \to g$ a.s.
- $g_n \geq 0$ and $|f_n| \leq g_n$ a.s.
- $\int g_n \, d\mu \to \int g \, d\mu < \infty$.

Then $f \in L^1$, and $\int f_n \, d\mu \to \int f \, d\mu$.

(1) $f_n, g_n, g \in L^1$ (3) $g_n \geq 0$ and $|f_n| \leq g_n$ a.s.

(2) $f_n \to f$ a.s. and $g_n \to g$ a.s. (4) $\int g_n \, d\mu \to \int g \, d\mu < \infty$.

$$\Rightarrow \quad f \in L^1, \text{ and } \int f_n \, d\mu \to \int f \, d\mu.$$

Pf. $|f| = \lim_{n\to\infty} |f_n| \leq \lim_{n\to\infty} |g_n| = |g|$ a.s. so $\int |f| \leq \int |g| < \infty$

$$\int (g \pm f) = \int \liminf_{n\to\infty} (g_n \pm f_n) \underset{\text{[Fatou]}}{\leq} \liminf_{n\to\infty} \int (g_n \pm f_n) = \lim_{n\to\infty} \int g_n + \liminf_{n\to\infty} \left(\pm \int f_n \right).$$

$$\therefore \int g \pm \int f \leq \int g + \begin{cases} \liminf_{n\to\infty} \int f_n \\ -\limsup_{n\to\infty} \int f_n \end{cases}$$