The Lebesgue Integral for $\overline{\mathbb{R}}$-Valued Functions

Let $f:(\Omega,\mathcal{F},\mu)\to(\overline{\mathbb{R}},\mathcal{B}(\overline{\mathbb{R}}))$ be measurable.

$$f_+ = f\,\mathbb{1}_{\{f\geq 0\}} \qquad f_- = -f\,\mathbb{1}_{\{f\leq 0\}} \qquad f = f_+ - f_-$$

Def: f is called integrable, $f\in\mathcal{L}^1(\Omega,\mathcal{F},\mu)$, if

$$(\star)\qquad \int f_\pm\,d\mu < \infty.$$

In this case, we define

$$\int f\,d\mu := \int f_+\,d\mu - \int f_-\,d\mu$$

Note: since $|f| = f_+ + f_-$, alternatively we have

$$\mathcal{L}^1(\Omega,\mathcal{F},\mu) = \left\{ f:\Omega\to\overline{\mathbb{R}} \text{ meas. s.t. } \int |f|\,d\mu < \infty \right\}$$

Note also: b/c $(\star)$, $f_\pm < \infty$ a.s. Therefore, we can restrict our attention to the complement of a nullset and assume f is $\mathbb{R}$-valued.

Proposition: [10.20] $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ is a $\mathbb{R}$-vector space,

$\int : \mathcal{L}^1(\Omega, \mathcal{F}, \mu) \to \mathbb{R}$ is linear, and

$f \leq g$ a.s. $\Rightarrow \int f \, d\mu \leq \int g \, d\mu \quad \forall f, g \in \mathcal{L}^1$.

Pf. $f, g \in \mathcal{L}^1$, $\underbrace{|af+bg|}_{\in L^+} \leq \underbrace{|a||f|}_{\in L^+} + \underbrace{|b||g|}_{\in L^+}$.

$\therefore \underbrace{\int |af+bg| \, d\mu}_{<\infty} \leq \int (|a||f| + |b||g|) \, d\mu = |a| \underbrace{\int |f| \, d\mu}_{<\infty} + |b| \underbrace{\int |g| \, d\mu}_{<\infty}$

$\therefore$

$(a>0)$

$$\int af \, d\mu := \int (af)_+ \, d\mu - \int (af)_- \, d\mu$$

$$= \int a f_+ \, d\mu - \int a f_- \, d\mu = a \int f_+ \, d\mu - a \int f_- \, d\mu$$

$$= a \left(\int f_+ - \int f_- \right)$$

$$= a \int f.$$

$$\int (f+g) = \int (f_+ - f_- + g_+ - g_-) = \int (f_+ + g_+) - \int (f_- + g_-)$$

$$= \int f_+ + \int g_+ - \int f_- - \int g_- \quad \cdots \text{ recombine.}$$

$f \leq g$ a.s. $\Rightarrow \int f \, d\mu \leq \int g \, d\mu \quad \forall f, g \in \mathcal{L}^1$.

$$f_+ - f_- = f \leq g = g_+ - g_- \Rightarrow f_+ + g_- \leq g_+ + f_-$$

$$\Rightarrow \int (f_+ + g_-) \leq \int (g_+ + f_-)$$

$$\int f_+ + \int g_- \leq \int g_+ + \int f_-$$

$$\therefore \underbrace{\int f_+ - \int f_-}_{= \int f} \leq \underbrace{\int g_+ - \int g_-}_{= \int g}.$$

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Bonus: $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$.

$$\left| \int f_+ - \int f_- \right| \leq \left| \int f_+ \right| + \left| \int f_- \right|$$

$$= \int f_+ + \int f_- = \int \underbrace{(f_+ + f_-)}_{|f|}$$

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Now we know how to "compute" expectations:

if $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ then $\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}$.

(As to how to actually compute it, we'll discuss that in the following lectures.)

L^1

Recall the pseudo-metric $d_\mu(A,B) = \mu^*(A \triangle B)$

if $A, B \in \mathcal{F}$: $= \mu(A \triangle B) = \int |\mathbb{1}_A - \mathbb{1}_B| \, d\mu$

We can think of $\mathcal{F}$ "$\subset$" $L^+(\mathcal{F})$ by $A \mapsto \mathbb{1}_A$.

This presents a natural way to extend d_μ to $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ "$\supset$" $\mu^{-1}([0,\infty))$

Def: The L^1-norm $\| \ \|_{L^1}$ is defined on $\mathcal{L}^1$ by

$$\|f\|_{L^1} := \int_\Omega |f| \, d\mu$$

$$d_\mu(A,B) = \int |\mathbb{1}_A - \mathbb{1}_B| \, d\mu = \|\mathbb{1}_A - \mathbb{1}_B\|_{L^1}$$

A norm on a vector space V is a function $\|\cdot\|: V \to [0,\infty)$ s.t.

- $\|\alpha f\| = |\alpha| \, \|f\| \quad \forall f \in V, \ \alpha \in \mathbb{R}$
- $\|f+g\| \le \|f\| + \|g\| \quad \forall f, g \in V$
- $\|f\| = 0$ if $\underbrace{\text{and only if}}$ $f = 0$.

if this part is missing, $\| \ \|$ is a seminorm.

If $\| \ \|$ is a seminorm, then $d(f,g) = \|f-g\|$ is a pseudo-metric.

$\| \ \|_{L^1}$ is a seminorm on $\mathcal{L}^1$.

$\|f\|_{L^1} = \int |f| \, d\mu \qquad \|f+g\|_{L^1} = \int \underset{\underset{|f|+|g|}{\wedge|}}{|f+g|} \, d\mu \le \int (|f|+|g|) \, d\mu = \int |f| \, d\mu + \int |g| \, d\mu$

$= \|f\|_{L^1} + \|g\|_{L^1}$

But it is not a norm.

[10.11]

$\|f\|_{L^1} = 0 \iff \int |f| \, d\mu = 0 \iff |f| = 0 \text{ a.s.}[\mu] \iff f = 0 \text{ a.s.}[\mu].$

Def: The relation $f \sim_\mu g$ iff $f = g$ a.s.$[\mu]$ is an equivalence relation on $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$.

$$L^1(\Omega, \mathcal{F}, \mu) := \mathcal{L}^1(\Omega, \mathcal{F}, \mu) / \sim_\mu$$

$$\left.\begin{aligned} [f]+[g] &:= [f+g] \\ \alpha[f] &:= [\alpha f] \end{aligned}\right\} \text{ well-defined.}$$

Elements are equivalence classes $[f]_{\sim_\mu}$ s.t. $f_1, f_2 \in [f]_{\sim_\mu} \Leftrightarrow f = g$ a.s.$[\mu]$.

Given $[f] \in L^1(\Omega, \mathcal{F}, \mu)$, a function $f_1 \in [f]$ is called a version of $[f]$.

Note: if $f_1 \sim_\mu f_2$, $\int f_1 \, d\mu = \int f_2 \, d\mu$. Thus, $\int [f] \, d\mu := \int f_1 \, d\mu$

[10.11]

for any $f_1 \in [f]$ makes sense.

All the properties of $\int \cdot \, d\mu$ on $\mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ descend nicely to $L^1(\Omega, \mathcal{F}, \mu)$.

And, even better:

$\| \ \|_{L^1}$ is a genuine norm on L^1.

$\therefore$ $d_\mu([f],[g]) = \| [f]-[g] \|_{L^1} = \| [f-g] \|_{L^1} = \int |f-g| \, d\mu$ is a genuine metric on L^1.

Going forward: we forget $\mathcal{L}^1$,
and treat the elements of L^1 as functions.

(Just keep in mind the "versions" business.)

We can also define L^p-norms: for $1 \le p < \infty$,

$$L^p(\Omega, \mathcal{F}, \mu) = (\sim_\mu \text{ equiv. classes of}) \{\text{meas. } f : \Omega \to \mathbb{R} \text{ s.t. } \int |f|^p < \infty\}$$

$$\|f\|_{L^p} := \left(\int |f|^p \, d\mu\right)^{1/p}.$$

Theorem: [17.27] $L^p(\Omega, \mathcal{F}, \mu)$ is a complete normed space.

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p} \qquad \text{(Minkowski's inequality)}$$

We will come back to L^p and completeness (and extending to $p=\infty$)
later when we talk about different modes of convergence of random variables.

For L^+, we have two integral convergence results: the MCTheorem, and Fatou's Lemma.
(Both can be partially extended to L^1, but they are not sufficiently powerful for most applications.)

The Dominated Convergence Theorem (DCT) [10.28]

Suppose $f_n, g_n, g \in L^1$, with

- $f_n \to f$ a.s. and $g_n \to g$ a.s.
- $g_n \geq 0$ and $|f_n| \leq g_n$ a.s.
- $\int g_n \, d\mu \to \int g \, d\mu < \infty$.

Then $f \in L^1$, and $\int f_n \, d\mu \to \int f \, d\mu$.

Typical statement: $g_n \equiv g \ \forall n$.
$|f_n| \leq g \in L^1$
$f_n \to f$ a.s. $\Rightarrow f \in L^1, \int f_n \to \int f$.

Special case: $\mu(\Omega) < \infty$.
Const.s $M < \infty$ are in L^1.
$\int M \, d\mu = M \mu(\Omega) < \infty$.

Bounded conv. thm: $(\mu(\Omega) < \infty)$.
$f_n \in L^1$, $|f_n| \leq M$
$f_n \to f$ a.s. $\Rightarrow |f| \leq M$ a.s. & $\int f_n \, d\mu \to \int f \, d\mu$

(1) $f_n, g_n, g \in L^1$ (3) $g_n \geq 0$ and $|f_n| \leq g_n$ a.s.

(2) $f_n \longrightarrow f$ a.s. and $g_n \longrightarrow g$ a.s. (4) $\int g_n \, d\mu \longrightarrow \int g \, d\mu < \infty$.

$$\Longrightarrow \quad f \in L^1, \quad \text{and} \quad \int f_n \, d\mu \longrightarrow \int f \, d\mu.$$

Pf. $|f| \overset{(2)}{=} \lim_{n\to\infty} |f_n| \overset{(3)}{\leq} \lim_{n\to\infty} |g_n| \overset{(2)}{=} |g|$ a.s. so $\int |f| \leq \int |g| \overset{(1)}{<} \infty$ $\therefore f \in L^1$.

$$\int (g \pm f) = \int \liminf_{n\to\infty} (g_n \pm f_n) \underset{[\text{Fatou}]}{\leq} \liminf_{n\to\infty} \int (g_n \pm f_n) = \underbrace{\lim_{n\to\infty} \int g_n}_{\overset{(4)}{=} \int g} + \liminf_{n\to\infty} \left(\pm \int f_n \right).$$

$$\therefore \cancel{\int g} \pm \int f \leq \cancel{\int g} + \begin{cases} \liminf_{n\to\infty} \int f_n \\ - \limsup_{n\to\infty} \int f_n \end{cases}$$

$$\therefore -\int f \qquad \limsup_{n\to\infty} \int f_n \leqq \int f \leqq \liminf_{n\to\infty} \int f_n \leqq \limsup_{n\to\infty} \int f_n$$

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