Last time we saw the core properties of the Lebesgue integral (on L^+). Amongst them:

$$f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$$

One quick and extremely useful corollary:

Markov's Inequality

If $f \geq 0$ and $\varepsilon, p > 0$, then

$$\mu\{f \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \int f^p \, d\mu .$$

Pf.

The Lebesgue Integral & Null Sets [Driver, § 10.1]

Notation: If $f \in L^+(\Omega, \mathcal{F})$ and $A \in \mathcal{F}$, then

$$\int_A f \, d\mu :=$$

Lemma: If $\varphi \in S^+_{\mathcal{F}}$ and $E \in \mathcal{F}$ s.t. $\mu(E) = 0$, then

$$\int_E \varphi \, d\mu = 0 \quad \text{and} \quad \therefore \int \varphi \, d\mu = \int_{E^c} \varphi \, d\mu .$$

Pf. Let $\varphi = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j}$.

Notation: In a measure space $(\Omega, \mathcal{F}, \mu)$,
a statement is true **almost everywhere (a.e.)**
or **almost surely (a.s.)**
if it holds true on a set $E \in \mathcal{F}$ s.t. $\mu(E^c) = 0$.

Prop. [10.11] Let $f, g \in L^+$.

1. If $f \le g$ a.s. then $\int f \, d\mu \le \int g \, d\mu$.
2. If $f = g$ a.s. then $\int f \, d\mu = \int g \, d\mu$.
3. If $\int f \, d\mu = 0$ then $f = 0$ a.s.

Pf. 1. Let $E = \{ f > g \}$, so $\mu(E) = 0$. Let $\varphi \in S_{\mathcal{F}}$, $\varphi \le f$.

Cor: If $f_n, f \in L^+$ satisfy $f_n \uparrow f$ a.s. then $\int f_n \uparrow \int f$.

Pf. By assumption, $\exists$ null set E s.t. $f_n \mathbb{1}_{E^c} \uparrow f \mathbb{1}_{E^c}$.

Fatou's Lemma

If $f_n \in L^+$, then $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Pf.

Some Applications

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space,

$\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ – not disjoint, but disjoint a.s.:

$$\mu(A_n \cap A_m) = 0 \quad \forall\, n \neq m$$

Then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

In a measure space $(\Omega, \mathcal{F}, \mu)$, with $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F}$, define $\{A_n \text{ i.o.}\} := \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$

$$= \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n$$

The Borel-Cantelli Lemma (I)

If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu\{A_n \text{ i.o.}\} = 0$.

Pf.

E.g. Toss a sequence of biased coins, where the probability of heads on the n^{th} toss is p_n.

Let X_n be the indicator of heads on the n^{th} toss.

$$\mathbb{P}(X_n = 1) = p_n \quad \mathbb{P}(X_n = 0) = 1 - p_n$$

$\therefore \mathbb{P}\{\text{heads i.o.}\} = \mathbb{P}\{X_n = 1 \text{ i.o.}\}$

So, if $\sum_{n=1}^{\infty} p_n < \infty$, then

What if $p_n = \frac{1}{n}$? Still decays to 0, but $\sum_{n=1}^{\infty} p_n = \infty$.

(wait for Borel-Cantelli II)