Last time we saw the core properties of the Lebesgue integral (on L^+). Amongst them:

$$f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$$

One quick and extremely useful corollary:

Markov's Inequality

If $f \geq 0$ and $\varepsilon, p > 0$, then

$$\mu\{f \geq \varepsilon\} \leq \frac{1}{\varepsilon^p} \int f^p \, d\mu .$$

Pf. $\mathbb{1}_{\{f \geq \varepsilon\}} \leq \mathbb{1}_{\{f \geq \varepsilon\}} \left(\frac{f}{\varepsilon}\right)^p \leq \left(\frac{f}{\varepsilon}\right)^p$.

$$\therefore \mu\{f \geq \varepsilon\} = \int \mathbb{1}_{\{f \geq \varepsilon\}} \, d\mu \overset{\downarrow}{\leq} \int \left(\frac{f}{\varepsilon}\right)^p d\mu = \frac{1}{\varepsilon^p} \int f^p \, d\mu .$$

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The Lebesgue Integral & Null Sets [Driver, § 10.1]

Notation: If $f\in L^+(\Omega,\mathcal{F})$ and $A\in\mathcal{F}$, then

$$\int_A f\,d\mu := \int_\Omega f\mathbb{1}_A\,d\mu = \sup\Big\{\int \underset{\varphi \rightsquigarrow \varphi\mathbb{1}_A}{\varphi\,d\mu : \varphi\in S_{\mathcal{F}}}, \varphi\le f\mathbb{1}_A\Big\}$$

Lemma: If $\varphi\in S_{\mathcal{F}}^+$ and $E\in\mathcal{F}$ s.t. $\mu(E)=0$, then

$$\int_E\varphi\,d\mu=0 \quad \text{and} \therefore \int\varphi\,d\mu=\int_{E^c}\varphi\,d\mu.$$

Pf. Let $\varphi=\sum_{j=1}^n \alpha_j\mathbb{1}_{A_j}$. Then $\varphi\mathbb{1}_E=\sum_{j=1}^n\alpha_j\mathbb{1}_{A_j}\mathbb{1}_E=\sum_{j=1}^n\alpha_j\mathbb{1}_{\underbrace{A_j\cap E}_{\text{still disjoint.}}}$

$$\therefore \int_E\varphi\,d\mu=\sum_{j=1}^n\alpha_j\underbrace{\mu(\cancel{A_j\cap E})}_{0}=0.$$

$$\therefore \int\varphi\,d\mu=\int\varphi(\underbrace{\mathbb{1}_E+\mathbb{1}_{E^c}}_{1})\,d\mu=\underbrace{\cancel{\int\varphi\mathbb{1}_E\,d\mu}}_{0}+\int\varphi\mathbb{1}_{E^c}\,d\mu$$

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Notation: In a measure space $(\Omega, \mathcal{F}, \mu)$,
a statement is true almost everywhere (a.e.)
or almost surely (a.s.)
if it holds true on a set $E \in \mathcal{F}$ s.t. $\mu(E^c) = 0$.

Prop. [10.11] Let $f, g \in L^+$.

1. If $f \leq g$ a.s.[μ] then $\int f \, d\mu \leq \int g \, d\mu$. ✓
2. If $f = g$ a.s.[μ] then $\int f \, d\mu = \int g \, d\mu$. ✓
3. If $\int f \, d\mu = 0$ then $f = 0$ a.s.

Pf. 1. Let $E = \{f > g\}$, so $\mu(E) = 0$. Let $\varphi \in S_{\mathcal{F}}$, $\varphi \leq f$.

$$\mathbb{1}_{E^c} \varphi \leq \mathbb{1}_{E^c} f \leq \mathbb{1}_{E^c} g \leq g. \Rightarrow \left.\underset{\int \varphi}{\underset{\shortparallel}{\int \mathbb{1}_{E^c} \varphi}} \leq \int g \right\} \therefore \underset{\sup\{\int \varphi \ldots\}}{\underset{\shortparallel}{\int f}} \leq \int g.$$

3. $\mu\{f \geq \frac{1}{n}\} \leq n \int f \, d\mu = 0$. $\forall n \in \mathbb{N}$. But $\{f \geq \frac{1}{n}\} \uparrow \{f > 0\}$

$$\therefore \underset{0}{\underset{\shortparallel}{\mu\{f \geq \tfrac{1}{n}\}}} \uparrow \mu\{f > 0\} \to 0.$$

Cor: If $f_n, f \in L^+$ satisfy $f_n \uparrow f$ a.s. then $\int f_n \uparrow \int f$.

Pf. By assumption, $\exists$ null set E s.t. $f_n \mathbb{1}_{E^c} \uparrow f \mathbb{1}_{E^c}$.

$$\therefore \int f_n\,d\mu \underset{10.11}{=} \int f_n \mathbb{1}_{E^c}\,d\mu \underset{MCT}{\uparrow} \int f \mathbb{1}_{E^c}\,d\mu \underset{10.11}{=} \int f\,d\mu$$

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Fatou's Lemma

If $f_n \in L^+$, then $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

Pf.

$$\liminf_{n\to\infty} f_n = \lim_{k\to\infty} \underbrace{\inf_{n\geq k} f_n}_{g_k} \qquad g_k \uparrow$$

Now: $g_k \leq f_n$ if $k \leq n$

$\therefore \int g_k \leq \int f_n$ if $k \leq n$.

$\therefore \int g_k \leq \liminf_{n\to\infty} \int f_n$.

$$\int \liminf_{n\to\infty} f_n = \int \lim_{k\to\infty} g_k \underset{MCT}{=} \lim_{k\to\infty} \int g_k$$

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Some Applications

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space,

$\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F}$ — not disjoint, but disjoint a.s.:

$$\mu(A_n \cap A_m) = 0 \qquad \forall\, n \neq m$$

Then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

$$\overset{\shortparallel}{\int \mathbb{1}_{\bigcup_{n=1}^{\infty} A_n}\, d\mu} \qquad \overset{\shortparallel}{\sum_{n=1}^{\infty} \int \mathbb{1}_{A_n}\, d\mu} = \int \sum_{n=1}^{\infty} \mathbb{1}_{A_n}\, d\mu$$

$\mathbb{1}_{\bigcup_{n=1}^{\infty} A_n}(\omega) = 1$ iff $\exists n$ s.t. $\omega \in A_n$

0 otherwise.

$\updownarrow$

$\sum_{n=1}^{\infty} \mathbb{1}_{A_n}(\omega) = \underline{\# n}$ s.t. $\omega \in A_n$ $\neq \mathbb{1}_{\bigcup_{n=1}^{\infty} A_n}$ iff $\underbrace{\exists n \neq m \text{ s.t. } \omega \in A_n \cap A_m}$.

$\uparrow$

$\mu = 0$.

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In a measure space $(\Omega, \mathcal{F}, \mu)$, with $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{F}$,

define $\{A_n \text{ i.o.}\} := \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$

$$= \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n = \limsup_{n \to \infty} A_n$$

The Borel-Cantelli Lemma (I)

If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu\{A_n \text{ i.o.}\} = 0$.

Pf. $\{A_n \text{ i.o.}\} = \{\omega \in \Omega : \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(\omega) = \infty\}$

By assumption, $\infty > \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \int \mathbb{1}_{A_n} d\mu = \int \sum_{n=1}^{\infty} \mathbb{1}_{A_n} d\mu$

$$\geq \int_{\{A_n \text{ i.o.}\}} \sum_{n=1}^{\infty} \mathbb{1}_{A_n} d\mu$$

$$\geq \int_{\{A_n \text{ i.o.}\}} M \, d\mu \qquad \forall M > 0$$

$$= M \mu\{A_n \text{ i.o.}\}$$

$$\therefore \mu\{A_n \text{ i.o.}\} = 0 \quad //$$

Eg. Toss a sequence of biased coins, where the probability of heads on the n^{th} toss is p_n.

Let X_n be the indicator of heads on the n^{th} toss.

$$\mathbb{P}(X_n = 1) = p_n \quad \mathbb{P}(X_n = 0) = 1 - p_n$$

So, if $\sum_{n=1}^{\infty} p_n < \infty$, then

$$\therefore \mathbb{P}\{\text{heads i.o.}\} = \mathbb{P}\{X_n = 1 \text{ i.o.}\} = 0.$$

Eg. $p_n \sim \frac{1}{n^2}$. Eventually, you'll never see heads.

What if $p_n = \frac{1}{n}$? Still decays to 0, but $\sum_{n=1}^{\infty} p_n = \infty$.

(wait for Borel-Cantelli II)