Integrating Non-Negative Measurable Functions [Driver, § 10.1]

$(\Omega, \mathcal{F}, \mu) \rightsquigarrow L^+ = L^+(\mathcal{F}) = \{ f : \Omega \to [0,\infty),\ f \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \}$

Def [10.1] For $f \in L^+$, the **Lebesgue integral** is:

$$\int f \, d\mu = \int_\Omega f(\omega)\, \mu(d\omega) := \sup \left\{ \int \varphi \, d\mu : \varphi \leq f,\ \varphi \text{ simple, measurable} \right\}$$

If μ is a probability measure, also denote it $\mathbb{E}[f] = \mathbb{E}_\mu[f]$.

Note: $\int f \, d\mu \in [0, \infty]$

Prop:
1. If $f \in L^+$, $\alpha \geq 0$, $\int \alpha f \, d\mu = \alpha \int f \, d\mu$
2. If $f, g \in L^+$, $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$
3. If $0 \leq f \leq g$, $\int f \, d\mu \leq \int g \, d\mu$.

1. $\int \alpha f \, d\mu = \alpha \int f \, d\mu , \quad \alpha \geq 0$

3. $f \leq g \in L^+ \Rightarrow \int f \, d\mu \leq \int g \, d\mu$

Before proceeding to additivity, we need a core robustness result for the Lebesgue integral.

Monotone Convergence Theorem [10.4]

If $f_n \in L^+$, $f_n \uparrow f$, then $\int f_n \, d\mu \uparrow \int f \, d\mu$.

Pf.

We can use the MCTheorem to give an explicit limit definition of $\int f\,d\mu$ for $f \in L^+$:

$$\varphi_n := \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{\{\frac{k}{2^n} < f \leq \frac{k+1}{2^n}\}} + 2^n \mathbb{1}_{\{f > 2^n\}} \in S_{\mathcal{F}}$$

$\varphi_n \uparrow f$ (even uniformly on $f^{-1}[-M,M]$)

$\therefore$ by MCTheorem, $\int f\,d\mu = \lim_{n\to\infty} \int \varphi_n\,d\mu$. I.e.

$$\int f\,d\mu = \lim_{n\to\infty}\left[\sum_{k=0}^{2^{2n}-1} \frac{k}{2^n}\,\mu\left\{\frac{k}{2^n} < f \leq \frac{k+1}{2^n}\right\} + 2^n \mu\{f > 2^n\}\right]$$

Note: This limit can be $+\infty$.
It always exists in $[0,\infty]$
since it is a limit of a
non-decreasing sequence in $[0,\infty]$.

Additivity: 2. $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$ for $f, g \in L^+$.

Bonus: If $f_n \in L^+$, then $\int \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu$.

Example. (Discrete measures)

Let $\rho : \Omega \to [0,\infty]$. Define μ on 2^{Ω} by

$$\mu = \sum_{\omega \in \Omega} \rho(\omega)\, \delta_{\omega}$$

For example, select $\{\omega_n\}_{n=1}^{\infty}$ in Ω, and let

$$\mu = \sum_{n=1}^{\infty} \rho_n \delta_{\omega_n} \quad \text{where } 0 \le \rho_n \le 1, \ \sum_{n=1}^{\infty} \rho_n = 1$$

Then μ is a discrete probability measure.

Then $\int f \, d\mu$

$$\mu = \sum_{\omega \in \Omega} \rho(\omega)\delta_\omega \quad \text{on } 2^\Omega .$$

We've shown that $\int f \, d\mu \leq \sum_\Omega f\rho$ for $f \in L^+$.

For the reverse ineq:

Fix an arbitrary finite set $\Lambda \subseteq \Omega$, and $N \in \mathbb{N}$.

$$\varphi_{N,\Lambda} := \mathbb{1}_\Lambda \min\{f, N\}$$