Integrating Non-Negative Measurable Functions [Driver, §10.1]

$(\Omega, \mathcal{F}, \mu) \rightsquigarrow L^+ = L^+(\mathcal{F}) = \{ f : \Omega \to [0,\infty),\ f \text{ is } \mathcal{F}/\mathcal{B}(\mathbb{R}) \text{ measurable} \}$

Def [10.1] For $f \in L^+$, the Lebesgue integral is:

$$\mu(f) = \int f \, d\mu = \int_\Omega f(\omega) \underbrace{\mu(d\omega)}_{d\mu(\omega)} := \sup\left\{ \int \varphi \, d\mu : \varphi \le f,\ \varphi \text{ simple, measurable} \right\}$$

If μ is a probability measure, also denote it $\mathbb{E}[f] = \mathbb{E}_\mu[f]$.

Can set $\varphi \geq 0$.

Note: $\int f \, d\mu \in [0, \infty]$

If $f \geq 0$, $\varphi \le f$
then $\varphi_+ \le f$

Prop: 1. If $f \in L^+$, $\alpha \geq 0$, $\int \alpha f \, d\mu = \alpha \int f \, d\mu$ $\quad (0 \cdot \infty = 0)$

2. If $f, g \in L^+$, $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

3. If $0 \le f \le g$, $\int f \, d\mu \le \int g \, d\mu$.

1. $\int \alpha f \, d\mu = \alpha \int f \, d\mu, \quad \alpha \geq 0$

If $\alpha = 0$, ✓

If $\alpha > 0$, $\int \alpha f = \sup\{\int \varphi : \varphi \in S_F, \ \varphi \leq \alpha f\}$

$= \sup\{\int \varphi : \varphi \in S_F, \ \underbrace{\varphi/\alpha}_{\psi} \leq f\}$

$= \sup\{\int \alpha \psi : \psi \in S_F \ \ \psi \leq f\}$

$= \alpha \cdot \sup\{\int \psi : \psi \in S_F \ \ \psi \leq f\}$

$= \alpha \int f.$

3. $f \leq g \in L^+ \Rightarrow \int f \, d\mu \leq \int g \, d\mu$

$$\int f \, d\mu = \sup \left\{ \int \varphi : \varphi \in S_{\mathcal{F}}, \ \varphi \leq f \right\}$$

$$\text{if } \varphi \leq f \leq g \Rightarrow \varphi \leq g$$

$$\sup \left\{ \int \varphi : \varphi \in S_{\mathcal{F}}, \ \varphi \leq f \right\}$$

$$\subseteq \left\{ \int \varphi : \varphi \in S_{\mathcal{F}}, \ \varphi \leq g \right\}$$

$$\leq \sup$$

Before proceeding to additivity, we need a core robustness result for the Lebesgue integral.

Monotone Convergence Theorem [10.4]

If $f_n \in L^+$, $f_n \uparrow f$, then $\int f_n d\mu \uparrow \int f d\mu$.

Pf. If $n \le m$, $f_n \le f_m \le f$

$\therefore \int f_n \le \int f_m \le \int f$

$\therefore \left(\int f_n\right)_{n=1}^{\infty} \uparrow$, $\therefore \lim_{n\to\infty} \int f_n \le \int f$.

For $(\ge)$, let $\varphi \in S_{\mathcal{F}}$, $0 \le \varphi \le f$. Fix $\alpha \in (0,1)$

$$\Omega_n := \{\omega \in \Omega : f_n(\omega) \ge \alpha\varphi(\omega)\} = (f_n - \alpha\varphi)^{-1}([0,\infty))$$

$\therefore \Omega_n \uparrow \Omega$ ($f \ge \varphi > \alpha\varphi$, $\therefore f_n(\omega) \ge \alpha\varphi(\omega)$ $\forall$ large $n = n(\omega)$)

By def: $f_n \ge \alpha \mathbb{1}_{\Omega_n} \cdot \varphi \in L^+$

$\therefore \int f_n \ge \int \alpha \mathbb{1}_{\Omega_n}\varphi = \alpha \int \mathbb{1}_{\Omega_n}\varphi$

$$\mathbb{1}_{\Omega_n}\varphi = \mathbb{1}_{\Omega_n} \sum_{t\ge 0} t \cdot \mathbb{1}_{\{\varphi = t\}} = \sum_{t\ge 0} t\, \mathbb{1}_{\{\varphi=t\}\cap\Omega_n}$$

$$\therefore \int \mathbb{1}_{\Omega_n}\varphi = \sum_{t\ge 0} t \cdot \underbrace{\mu(\{\varphi = t\}\cap\Omega_n)}$$

$\{\varphi = t\}\cap\Omega_n \uparrow \{\varphi = t\}$

$\xrightarrow{n\to\infty}$ $\mu\{\varphi = t\}$

$$\therefore \int \mathbb{1}_{\Omega_n}\varphi \xrightarrow[n\to\infty]{} \sum_{t\ge 0} t\cdot\mu\{\varphi=t\} = \int\varphi.$$

$$\therefore \lim_{n\to\infty}\int f_n \ge \alpha\int\varphi$$

$\downarrow 1$

$$\ge \int \varphi$$

$$\therefore \ge \sup\left\{\int\varphi : \varphi \le f\right\} = \int f$$

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We can use the MCTheorem to give an explicit limit definition of $\int f \, d\mu$ for $f \in L^+$:

$$\varphi_n := \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mathbb{1}_{\{\frac{k}{2^n} < f \leq \frac{k+1}{2^n}\}} + 2^n \mathbb{1}_{\{f > 2^n\}} \in S_{\mathcal{F}}$$

$\varphi_n \uparrow f$ (even uniformly on $f^{-1}[-M, M]$)

$\therefore$ by MCTheorem, $\int f \, d\mu = \lim_{n \to \infty} \int \varphi_n \, d\mu$. I.e.

$$\int f \, d\mu = \lim_{n \to \infty} \left[\sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mu\left\{\frac{k}{2^n} < f \leq \frac{k+1}{2^n}\right\} + 2^n \mu\{f > 2^n\} \right]$$

Note: This limit can be $+\infty$.
It always exists in $[0, \infty]$ since it is a limit of a non-decreasing sequence in $[0, \infty]$.

Additivity. 2. $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$ for $f, g \in L^+$.

Find $\varphi_n \in S_f$, $\varphi_n \uparrow f$

$\psi_n \in S_f$, $\psi_n \uparrow g$ $\Big\}$ $\varphi_n + \psi_n \uparrow (f+g)$

$\therefore \int \psi_n \to \int g$

$\int \varphi_n \to \int f$ $\Big\}$ $\therefore \int f + \int g \leftarrow \int \psi_n + \int \varphi_n = \int (\psi_n + \varphi_n) \to \int (f+g)$

Bonus: If $f_n \in L^+$, then $\int \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int f_n\,d\mu$.

Let $g_N = \sum_{n=1}^{N} f_n$ Since $f_n \geq 0$, $g_N \uparrow g = \sum_{n=1}^{\infty} f_n$

$\therefore$ MCT $\int g_N \uparrow \int g = \int \sum_{n=1}^{\infty} f_n$

$$\int \sum_{n=1}^{N} f_n = \sum_{n=1}^{N} \int f_n \longrightarrow \sum_{n=1}^{\infty} \int f_n$$

Example (Discrete measures)

Let $\rho : \Omega \to [0,\infty]$. Define μ on 2^Ω by

$$\mu = \sum_{\omega\in\Omega} \rho(\omega)\delta_\omega \qquad \mu(A) = \sum_{\omega\in A} \rho(\omega) := \sup_{\substack{\Lambda\subseteq A \\ \#\Lambda<\infty}} \sum_{\omega\in\Lambda} \rho(\omega).$$

For example, select $\{\omega_n\}_{n=1}^\infty$ in Ω, and let

$$\mu = \sum_{n=1}^\infty \rho_n \delta_{\omega_n} \quad \text{where } 0\le \rho_n \le 1,\ \sum_{n=1}^\infty \rho_n = 1$$

Then μ is a discrete probability measure.

Then $\int f d\mu = \sum_{\omega\in\Omega} f(\omega)\rho(\omega) = \sum_\Omega f\rho.$

Pf. If $\varphi \ge 0$, $\varphi \in S_{2^\Omega}$, $\varphi = \sum_{t\ge 0} t\cdot \mathbb{1}_{\{\varphi=t\}}$

$$\therefore \int \varphi\, d\mu = \sum_{t\ge0} t\cdot \mu\{\varphi=t\} = \sum_{t\ge0} t\cdot \sum_{\omega\in\Omega}\rho(\omega)\mathbb{1}_{\{\varphi(\omega)=t\}}$$
$$= \sum_{\omega\in\Omega}\rho(\omega)\sum_{t\ge0} t\cdot \mathbb{1}_{\{\varphi(\omega)=t\}}$$
$$= \sum_{\omega\in\Omega}\rho(\omega)\varphi(\omega).$$

$\therefore$ If $\varphi\le f$,

$$\int\varphi = \sum_\Omega \varphi\rho \le \sum_\Omega f\rho$$

Take $\sup_\varphi$ $\uparrow$

$$\therefore \int f \le \sum_\Omega f\rho.$$

$\mu = \sum_{\omega \in \Omega} \rho(\omega) \delta_\omega$ on 2^Ω.

We've shown that $\int f \, d\mu \leq \sum_\Omega f\rho$ for $f \in L^+$.

For the reverse ineq:

Fix an arbitrary finite set $\Lambda \subseteq \Omega$, and $N \in \mathbb{N}$.

$$\varphi_{N,\Lambda} := \mathbb{1}_\Lambda \min\{f, N\} \leq f$$

$$\sum_\Omega \varphi_{N,\Lambda} \rho = \int \varphi_{N,\Lambda} \, d\mu \leq \int f \, d\mu \quad \forall N, \Lambda$$

$\varphi_{N,\Lambda} \uparrow \mathbb{1}_\Lambda f$ as $N \to \infty$ $\therefore$ $\sum_\Omega \mathbb{1}_\Lambda f \rho \leq \int f \, d\mu$.

$$\sum_\Omega \mathbb{1}_\Lambda f \rho = \sum_{\omega \in \Lambda} f(\omega) \rho(\omega)$$

$$\sum_{\omega \in \Omega} f(\omega)\rho(\omega) =: \sup_{\substack{\Lambda \subseteq \Omega \\ \#\Lambda < \infty}} \sum_{\omega \in \Lambda} f(\omega)\rho(\omega) \leq \int f \, d\mu$$

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