Simple Integration [Driver, § 5.5]

$(\Omega, \mathcal{F}, \mathbb{P})$

↑

finite $\therefore \quad \mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\{\omega\})$

Def. General measure space $(\Omega, \mathcal{F}, \mu)$ and $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable simple function $f = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j}$

Define $\int f \, d\mu :=$

If μ is a probability measure, often denoted as $\mathbb{E}(f) = \mathbb{E}_\mu(f)$.

E.g. $\mathbb{E}(\mathbb{1}_A)$

$\uparrow$

$A \in \mathcal{F}$, $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ over $(\Omega, \mathcal{F}, \mathbb{P})$

Proposition: [5.27]

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let
$S_{\mathcal{F}} = \{\text{simple } \mathcal{F}/\mathcal{B}(\mathbb{R})\text{-measurable functions}\}$

Then $S_{\mathcal{F}}$ is a real vector space, and

$\int \cdot \, d\mu : S_{\mathcal{F}} \to \mathbb{R}$ is a positive linear functional

I.e. * $f, g \in S_{\mathcal{F}}$, $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha f + \beta g \in S_{\mathcal{F}}$

& $\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu$

* If $f \leq g \in S_{\mathcal{F}}$ then $\int f\, d\mu \leq \int g\, d\mu$

(In particular $f \geq 0 \Rightarrow \int f\, d\mu \geq 0$)

Pf. 1. $f \in S_{\mathcal{F}}$, $\beta \in \mathbb{R} \Rightarrow \beta f \in S_{\mathcal{F}}$, $\int \beta f \, d\mu = \beta \int f \, d\mu$

2. $f, g \in S_{\mathcal{F}} \Rightarrow f + g \in S_{\mathcal{F}}$

3. $f, g \in S_{\mathcal{F}} \Rightarrow \int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

4. $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$.

Special Bonus:

5. $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$.

Application: Inclusion - Exclusion

$(\Omega, \mathcal{F}, \mu)$ measure space, $A_1, \dots, A_n \in \Omega$ with $\mu(A_j) < \infty$ $\forall j$.

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \le k_1 < \dots < k_j \le n} \mu(A_{k_1} \cap \dots \cap A_{k_j})$$

E.g. $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2)$

$\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3) - \mu(A_1 \cap A_2) - \mu(A_1 \cap A_3) - \mu(A_2 \cap A_3) + \mu(A_1 \cap A_2 \cap A_3)$

Pf. $A = A_1 \cup \dots \cup A_n$. $A^c = A_1^c \cap \dots \cap A_n^c$.

$\mathbb{1}_{A^c} = \mathbb{1}_{A_1^c} \cdot \mathbb{1}_{A_2^c} \cdot \dots \cdot \mathbb{1}_{A_n^c} = (1 - \mathbb{1}_{A_1})(1 - \mathbb{1}_{A_2}) \cdots (1 - \mathbb{1}_{A_n})$

Example. Shuffle a deck of n cards. What is the probability that at least one card remains in the same position after the shuffle? What is the expected number of unmoved cards?

$\Omega = S_n$ permutations of $\{1, \dots, n\}$ $\quad \mathcal{F} = 2^{\Omega} \quad \mathbb{P}(A) = \#A / n!$, uniformly random permutations

$A_i = \{\omega \in \Omega : \omega(i) = i\}$ the set of permutations fixing the i^{th} card.

$B = \bigcup_{i=1}^{n} A_i \qquad \mathbb{P}(B)$

What about the expected number X of fixed cards?

$X =$