Simple Integration [Driver, § 5.5]

$(\Omega, \mathcal{F}, \mathbb{P})$

finite

$\therefore \quad \mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \mathbb{P}(\{\omega\}) = \sum_{t \in \mathbb{R}} t \cdot \underbrace{\mathbb{P}(X = t)}_{\mathbb{P}\{\omega \in \Omega : X(\omega) = t\}} = \sum_{j=1}^{n} \alpha_j \, \mathbb{P}(X = \alpha_j)$

takes only finitely-many values $\{\alpha_j\}_{j=1}^{n}$

Def General measure space $(\Omega, \mathcal{F}, \mu)$ and $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable simple function $f = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j} \leftarrow A_j \in \mathcal{F}$

Define $\int f \, d\mu := \sum_{t \in \mathbb{R}} t \cdot \mu\{f = t\} = \sum_{j=1}^{n} \alpha_j \, \mu(A_j)$

If μ is a probability measure, often denoted as $\mathbb{E}(f) = \mathbb{E}_\mu(f)$.

E.g. $\mathbb{E}(\mathbb{1}_A) = 0 \cdot \mathbb{P}(A^c) + 1 \cdot \mathbb{P}(A) = \mathbb{P}(A)$

$A \in \mathcal{F}$, $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ over $(\Omega, \mathcal{F}, \mathbb{P})$

Proposition: [5.27]

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let
$S_{\mathcal{F}} = \{$simple $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable functions$\}$

Then $S_{\mathcal{F}}$ is a real vector space, and

$\int \cdot \, d\mu : S_{\mathcal{F}} \to \mathbb{R}$ is a positive linear functional

I.e. * $f, g \in S_{\mathcal{F}}$, $\alpha, \beta \in \mathbb{R} \Rightarrow \alpha f + \beta g \in S_{\mathcal{F}}$

& $\int (\alpha f + \beta g)\, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$

* If $f \leq g \in S_{\mathcal{F}}$ then $\int f \, d\mu \leq \int g \, d\mu$

(In particular $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$)

$\int g - \int f = \int \underset{\substack{\vee\!\!| \\ 0}}{(g - f)} \geq \int 0 = 0.$

Pf. 1. $f \in S_{\mathcal{F}}, \beta \in \mathbb{R} \Rightarrow \beta f \in S_{\mathcal{F}}, \int \beta f \, d\mu = \beta \int f \, d\mu$

$$f = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{A_j}, \quad \beta f = \sum_{j=1}^{n} \beta \alpha_j \mathbb{1}_{A_j}$$

$$\int \beta f \, d\mu = \sum_{j=1}^{n} \beta \alpha_j \mu(A_j) = \beta \sum_{j=1}^{n} \alpha_j \mu(A_j) = \beta \int f \, d\mu.$$

2. $f, g \in S_{\mathcal{F}} \Rightarrow f+g \in S_{\mathcal{F}}$

For any $t \in \mathbb{R}$, $\{f+g = t\} = \bigcup_{\substack{x, y \in \mathbb{R} \\ x+y=t}} \{f = x\} \cap \{g = y\}$

$= \phi$ for all but finitely many x, y

$\therefore \forall$ but finitely many t.

$\therefore f+g \in S_{\mathcal{F}}.$

3. $f, g \in S_{\mathcal{F}} \Rightarrow \int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

$$\int (f+g)\,d\mu = \sum_{t \in \mathbb{R}} t \cdot \mu\{f+g=t\}$$

$$= \sum_{t \in \mathbb{R}} \sum_{\substack{x,y \in \mathbb{R} \\ x+y=t}} t \cdot \mu(\{f=x\} \cap \{g=y\})$$

$t = x+y$

$$= \sum_{x,y \in \mathbb{R}} (x+y)\, \mu(\{f=x\} \cap \{g=y\})$$

$$= \sum_{x,y} x\, \underbrace{\mu(\{f=x\} \cap \{g=y\})} + \sum_{x,y} y\, \mu(\{f=x\} \cap \{g=y\})$$

$$= \sum_{x} x \sum_{y} \mu(\{f=x\} \cap \{g=y\})$$

$$\mu\Big(\underbrace{\bigsqcup_{y} \{f=x\} \cap \{g=y\}}\Big)$$

$$\sum_{x} x \cdot \mu\{f=x\} = \int f\,d\mu$$

$$\sum_{y} y \cdot \mu\{g=y\}$$

$$+ \int g\,d\mu.$$

4. $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$.

$$\int f\,d\mu = \sum_{t \in \mathbb{R}} t \cdot \mu\{f = t\} = \sum_{t \geq 0} t \, \mu\{f = t\} \geq 0.$$

$\mu\{f=t\} = \phi$ if $t < 0$

Special Bonus:

5. $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$.

$$\left|\int f\,d\mu\right| = \left| \sum_t t \cdot \mu\{f = t\} \right| \leq \sum_t \left| t \cdot \mu\{f = t\} \right| = \sum_t |t| \cdot \mu\{f = t\} = \int |f| \, d\mu.$$

Δ-ineq.

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Application: Inclusion - Exclusion

$\mathbb{1}_{A\cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$

$(\Omega, \mathcal{F}, \mu)$ measure space, $A_1, \dots, A_n \in \Omega$ with $\mu(A_j) < \infty \ \forall j$.

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n (-1)^{j+1} \sum_{1\le k_1 < \dots < k_j \le n} \mu(A_{k_1} \cap \dots \cap A_{k_j})$$

E.g. $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2)$

$$\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3) - \mu(A_1 \cap A_2) - \mu(A_1 \cap A_3) - \mu(A_2 \cap A_3) + \mu(A_1 \cap A_2 \cap A_3)$$

Pf. $A = A_1 \cup \dots \cup A_n$. $A^c = A_1^c \cap \dots \cap A_n^c$.

$$\underset{\substack{\parallel \\ 1 - \mathbb{1}_A}}{\mathbb{1}_{A^c}} = \mathbb{1}_{A_1^c} \cdot \mathbb{1}_{A_2^c} \cdots \mathbb{1}_{A_n^c} = (1 - \mathbb{1}_{A_1})(1 - \mathbb{1}_{A_2}) \cdots (1 - \mathbb{1}_{A_n})$$

$$= \sum_{j=0}^n \sum_{1 \le k_1 < \dots < k_j \le n} (-\mathbb{1}_{A_{k_1}}) \cdots (-\mathbb{1}_{A_{k_j}})$$

$$\boxed{\mathbb{1}_A = \sum_{j=1}^n \sum_{1 \le k_1 < \dots < k_j \le n} (-1)^{j+1} \mathbb{1}_{A_{k_1} \cap \dots \cap A_{k_j}}}$$

$$= \sum_{j=0}^n \sum_{k_1 < \dots < k_j} (-1)^j \mathbb{1}_{A_{k_1} \cap \dots \cap A_{k_j}}.$$

$$= 1 + \sum_{j=1}^n (\text{ " } \text{ " })$$

$$\int \cdot \, d\mu = \int (\quad) \, d\mu$$

Example. Shuffle a deck of n cards. What is the probability that at least one card remains in the same position after the shuffle? What is the expected number of unmoved cards?

$\Omega = S_n$ permutations of $\{1, \dots, n\}$ $\quad \mathcal{F} = 2^{\Omega}$ $\quad \mathbb{P}(A) = \#A / n!$ uniformly random permutations

$A_i = \{\omega \in \Omega : \omega(i) = i\}$ the set of permutations fixing the i^{th} card.

$$B = \bigcup_{i=1}^{n} A_i \qquad \mathbb{P}(B) = \sum_{i=1}^{n} (-1)^{i+1} \sum_{1 \le k_1 < \dots < k_i \le n} \mathbb{P}(A_{k_1} \cap \dots \cap A_{k_i})$$

$$\{\omega \in \Omega : \omega(k_1) = k_1, \dots, \omega(k_i) = k_i\} \sim S_{n-i}$$

$$\therefore \# = (n-i)! \quad \therefore \mathbb{P} = \frac{(n-i)!}{n!}$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \frac{\cancel{(n-i)!}}{\cancel{n!}} \underbrace{\#\{(k_1, \dots, k_i) \; 1 \le k_1 < \dots < k_i \le n\}}_{\binom{n}{i} = \frac{\cancel{n!}}{i!\cancel{(n-i)!}}}$$

$$= \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}$$

$$= 1 - \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

$$\xrightarrow{n \to \infty} 1 - \frac{1}{e} \doteq 63.2\%$$

What about the expected number X of fixed cards?

$$X = \sum_{i=1}^{n} \mathbb{1}_{A_i}$$

$$\therefore \ \mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{A_i}) = \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

$$\mathbb{P}\{\omega : \omega(i) = i\} = \frac{(n-1)!}{n!} = \frac{1}{n}$$